IMMERSIONS OF RP^{2^e-1}

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This paper is dedicated to Michael Barratt on the occasion of his 80th birthday.

Abstract. We prove that RP^{2^e-1} can be immersed in $\mathbb{R}^{2^{e+1}-e-8}$ provided $e \geq 7$. If $e \geq 14$, this is 2 better than previously known immersions. Our method is primarily an induction on geometric dimension, incorporating also sections obtained from the Radon-Hurwitz theorem.

1. Statement of result and background

Our main result is the following immersion theorem for real projective spaces.

Theorem 1.1. *If $e \geq 7$, then RP^{2^e-1} can be immersed in $\mathbb{R}^{2^{e+1}-e-8}$.*

This improves, in these cases, by 2 dimensions upon the result of Milgram ([9]), who proved, by constructing bilinear maps, that if $n \equiv 7 \bmod 8$, then RP^n can be immersed in $\mathbb{R}^{2n-\alpha(n)-4}$, where $\alpha(n)$ denotes the number of 1's in the binary expansion of n. In [2, 1.2], the first and third authors used obstruction theory to prove that if $n \equiv 7 \bmod 8$, then RP^n can be immersed in $\mathbb{R}^{2n-D}$, where $D = 14, 16, 17, 18$ if $\alpha(n) = 7, 8, 9, \geq 10$. That result, with $n = 2^e - 1$, is 1 or 2 dimensions stronger than ours for $7 \leq e \leq 11$. If $e \geq 13$, then our result improves on the result of [2] by $e - 12$ dimensions. Thus Theorem 1.1 improves on all known results by 2 dimensions if $e \geq 14$.

In [6], James proved that RP^{2^e-1} cannot be immersed in $\mathbb{R}^{2^{e+1}-2e-\delta}$ where $\delta = 3, 2, 2, 4$ for $e \equiv 0, 1, 2, 3 \bmod 4$. In [5], an immersion result for RP^{2^e-1} was announced in dimension 1 greater than that of James' nonimmersion, which would have been optimal. However, a mistake in the argument of [5] was pointed out by Crabb and Steer. We hope that a slight improvement in our argument might enable us to prove an

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immersion result in dimension 1 greater than that of James' nonimmersion (possibly 2 greater if $e \equiv 3 \mod 4$). We will point out in Remark 2.11 what would be required for this improvement.

2. Outline of proof

In this section we outline the proof of Theorem 1.1. In subsequent sections, we will fill in details.

If θ is a vector bundle over a compact connected space X, we define the geometric dimension of θ, denoted $\mathrm{gd}(\theta)$, to be the fiber dimension of θ minus the maximum number of linearly independent sections of θ. Equivalently, if $\dim(\theta) = n$, then $\mathrm{gd}(\theta)$ equals the smallest integer k such that the map $X \xrightarrow{\theta} BO(n)$ which classifies θ factors through $BO(k)$. The following lemma is standard (See e.g. [10, 4.2]). Here and throughout, ξ_n denotes the Hopf line bundle over RP^n.

Lemma 2.1. *Let $\phi(n)$ denote the number of positive integers i satisfying $i \le n$ and $i \equiv 0, 1, 2, 4 \mod 8$. Suppose $n > 8$. Then RP^n can be immersed in $\mathbb{R}^{n+k}$ if and only if $\mathrm{gd}((2^{\phi(n)} - n - 1)\xi_n) \le k$.*

Thus Theorem 1.1 will follow from the following result, to the proof of which the remainder of this paper will be devoted.

Theorem 2.2. *If $e \ge 7$, then $\mathrm{gd}((2^{2^{e-1}-1} - 2^e)\xi_{2^e-1}) \le 2^e - e - 7$.*

The bulk of the work toward proving Theorem 2.2 will be a determination of upper bounds for $\mathrm{gd}(2^e\xi_n)$ for all $n \equiv 7 \mod 8$ by induction on e. A similar method could be employed for all n, but we restrict to $n \equiv 7 \mod 8$ to simplify the already formidable arithmetic. We let $A_k = RP^{8k+7}$, and denote $\mathrm{gd}(m\xi_{8k+7})$ by $\mathrm{gd}(m, k)$.

The classifying map for $2^e\xi_{8k+7}$ will be viewed as the following composite.

$$A_k \xrightarrow{d} (A_k \times A_k)^{(8k+7)} \hookrightarrow \bigcup_j A_j \times A_{k-j} \xrightarrow{f\times f} BO_{2^{e-1}} \times BO_{2^{e-1}} \to BO_{2^e} \tag{2.3}$$

Here d is a cellular map homotopic to the diagonal map, $X^{(n)}$ denotes the n-skeleton of X, and f classifies $2^{e-1}\xi$. We write BO_m for $BO(m)$ for later notational convenience.

As a first step, we would like to use (2.3) to deduce that

$$\mathrm{gd}(2^e, k) \le \max\{\mathrm{gd}(2^{e-1}, j) + \mathrm{gd}(2^{e-1}, k-j) : 0 \le j \le k\}.$$

In order to make this deduction, we need to know that the liftings of the various $2^{e-1}\xi_{8j+7}$ to various BO_m have been made compatibly.

Definition 2.4. *If θ is a vector bundle over a filtered space $X_0 \subset \cdots \subset X_k$, we say that*

$$\mathrm{gd}(\theta|X_i) \le d_i \textit{ compatibly for } i \le k$$

if there is a commutative diagram

$$\begin{array}{ccccccccc}
X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_k & & \\
\downarrow & & \downarrow & & & & \downarrow & & \\
BO_{d_0} & \longrightarrow & BO_{d_1} & \longrightarrow & \cdots & \longrightarrow & BO_{d_k} & \longrightarrow & BO_{\dim(\theta)}
\end{array}$$

where the map $X_k \to BO_{\dim(\theta)}$ classifies θ, and the horizontal maps are the usual inclusions.

If $X_0 \subset \cdots \subset X_k$ and $Y_0 \subset \cdots \subset Y_k$ are filtered spaces, we define, for $0 \le i \le k$,

$$(X \times Y)_i := \bigcup_{j=0}^{i} X_j \times Y_{i-j}.$$

Then $(X \times Y)_0 \subset \cdots \subset (X \times Y)_k$ is clearly a filtered space. We will prove the following general result in Section 3.

Proposition 2.5. *Suppose* $\mathrm{gd}(\theta|X_i) \le d_i$ *compatibly for $i \le k$ and* $\mathrm{gd}(\eta|Y_i) \le d'_i$ *compatibly for $i \le k$. For $0 \le j \le k$, let $e_j = \max(d_i + d'_{j-i} : 0 \le i \le j)$. Then* $\mathrm{gd}(\theta\times\eta|(X\times Y)_j) \le e_j$ *compatibly for $j \le k$. Moreover, if $X = Y$ and $\theta = \eta$, then the maps $(X\times X)_j \xrightarrow{f} BO_{e_j}$ can be chosen to satisfy $f\circ T = f$, where $T : X\times X \to X\times X$ interchanges factors.*

We will begin an induction using some known compatible bounds for $\mathrm{gd}(16, i)$. Proposition 2.5 will, after restriction under the diagonal map, allow us to prove $\mathrm{gd}((\sum 2^{e_i})\xi_n) \le \max\{\sum \mathrm{gd}(2^{e_i}\xi_{m_i}) : \sum m_i = n\}$. These bounds are not yet strong enough to yield new immersion results. Next, we must improve the bounds by taking advantage of paired obstructions. The following result will be proved in Section 3.

Proposition 2.6. *Let $BO_n[\rho]$ denote the pullback of BO_n and the $(\rho-1)$-connected cover $BO[\rho]$ over BO, and let $s = \min(\rho + 2m - 1, 4m - 1)$.*

(1) *There are equivalences c_1' and c_2' such that the following diagram commutes.*

$$\begin{tikzcd}
BO_{2m}[\rho]^{(s)} \arrow[r, "q_1"] \arrow[d, "p_2"'] & (BO_{2m}[\rho]/BO_{2m-1}[\rho])^{(s)} \arrow[r, "c_1'"] \arrow[d, "p_2'"'] & S^{2m} \arrow[d, "i"'] \\
BO_{2m+1}[\rho]^{(s)} \arrow[r, "q_2"] & (BO_{2m+1}[\rho]/BO_{2m-1}[\rho])^{(s)} \arrow[r, "c_2'"] & \Sigma P_{2m-1}^{2m}.
\end{tikzcd}$$

Preparatory to the next two parts, we expand this diagram as follows, with $c_i = c_i' \circ q_i$ and (X, A) a finite CW pair.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "j"'] & BO_{2m-1}[\rho]^{(s)} \arrow[d, "p_1"'] & \\
X & BO_{2m}[\rho]^{(s)} \arrow[r, "c_1"] \arrow[d, "p_2"'] & S^{2m} \arrow[d, "i"'] \\
& BO_{2m+1}[\rho]^{(s)} \arrow[r, "c_2"] & \Sigma P_{2m-1}^{2m}.
\end{tikzcd}$$

(2) *Suppose* $\dim(X) < s$*, and we are given $X \xrightarrow{f} BO_{2m}[\rho]^{(s)}$ such that $f \circ j = p_1 \circ f_1$ and $c_1 \circ f$ factors as $X \to X/A \xrightarrow{g} S^{2m}$ with $[g]$ divisible by 2 in $[X/A, S^{2m}]$.*[1] *Then $p_2 \circ f$ lifts to a map $X \xrightarrow{\ell} BO_{2m-1}[\rho]^{(s)}$ whose restriction to A equals f_1.*

(3) *Suppose, on the other hand,* $\dim(X) \le s$*, and we are given $X \xrightarrow{f'} BO_{2m+1}[\rho]^{(s)}$ such that $f' \circ j = p_2 \circ p_1 \circ f_1$ and $c_2 \circ f'$ factors as $X \to X/A \xrightarrow{g'} \Sigma P_{2m-1}^{2m}$ with $[g']$ divisible by 2 in $[X/A, \Sigma P_{2m-1}^{2m}]$. Then f' is homotopic rel A to a map which lifts to $BO_{2m}[\rho]^{(s)}$.*

In Section 4, we will implement Propositions 2.5 and 2.6 to prove that the last part of the following important result follows from the first five parts, while in Section 5, we will establish the first five parts. Here and throughout, $\nu(-)$ denotes the exponent of 2 in an integer.

Theorem 2.7. *There is a function $g(e, k)$ defined for $e \ge 4$ and $k \ge 0$ satisfying:*

(1) *If $k \ge 2^{e-3}$, then $g(e, k) = 2^e$.*

[1] Note that $[X/A, S^{2m}]$ is in the stable range, from which it gets its group structure.

(2) *If* $e > 4k+2$, *then* $g(e,k) = 0$, *while if* $e \le 4k+2$ *and* $k > 1$, *then* $g(e,k) \ge 4k+4$.

(3) *If* $0 \le \ell \le k$, *then* $g(e,k) \ge g(e-1,\ell) + g(e-1,k-\ell) - 1$.

(4) *If* $[(e+1)/4] \le 2\ell < 2^{e-3}$, *then* $g(e,2\ell) \ge 2g(e-1,\ell)+1$.

(5) *Either* $g(e,k) = g(e,k-1)$ *or* $g(e,k) \ge g(e,k-1)+2$.

(6) $\mathrm{gd}(2^e,k) \le g(e,k)$ *compatibly for all* k.

By restricting the lifting of P^{8k+7} to P^{8k+i} for $0 \le i \le 6$, we may use this result to obtain compatible liftings of $2^e\xi_n$ for all n.

The function g will be semiexplicitly defined in (5.2), 5.3, and 5.4. In Table 2.8, we list its values for small values of the parameters. We prefer not to tabulate the values $g(e,k) = 2^e$ when $k > 2^{e-3}$. The numbers in boldface will be given special attention at the beginning of Section 5.

Table 2.8. Values of $g(e,k)$ when $e \le 15$ and $k \le 16$.

					k											
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
4	**7**	16														
5	6	**15**	**22**	32												
e 6	5	14	21	**31**	**37**	**46**	**53**	64								
7	0	13	20	30	36	45	52	**63**	**68**	**77**	**84**	**94**	**100**	**109**	**116**	128
8	0	12	19	29	35	44	51	62	67	76	83	93	99	108	115	**127**
9	0	12	18	28	34	43	50	61	66	75	82	92	98	107	114	126
10	0	12	17	27	33	42	49	60	65	74	81	91	97	106	113	125
11	0	0	16	26	32	41	48	59	64	73	80	90	96	105	112	124
12	0	0	16	25	31	40	47	58	63	72	79	89	95	104	111	123
13	0	0	16	24	30	39	46	57	62	71	78	88	94	103	110	122
14	0	0	16	23	29	38	45	56	61	70	77	87	93	102	109	121
15	0	0	0	22	28	37	44	55	60	69	76	86	92	101	108	120

To obtain the best results, we must insert one more bit of sectioning information—linear vector fields on S^n yield vector fields on P^n and hence sections of $(n+1)\xi_n = \tau(P^n) \oplus \epsilon$. Let

$$\rho(4a+b) = 8a + 2^b \text{ if } 0 \le b \le 3.$$

Eckmann ([4]) used the Radon-Hurwitz theorem to show that S^n has $\rho(\nu(n+1)) - 1$ linearly independent linear fields of tangent vectors and hence $(n+1)\xi_n$ has $\rho(\nu(n+1))$ linearly independent sections. We obtain the following well-known result.

Proposition 2.9. *For $e \geq 2$,* $\mathrm{gd}(2^e\xi_{2^e-1}) \leq 2^e - \rho(e)$.

If we wish to incorporate these into any subsequent induction argument, it is necessary that the liftings be compatible with the liftings already obtained on the lower skeleta. All we can easily assert is the following.

Proposition 2.10. *Let*

$$d_{e,n} = \begin{cases} 0 & \text{if } n \leq \rho(e) \\ g(e, [\frac{n}{8}]) & \text{if } \rho(e) < n < 2^e - \rho(e) \\ 2^e - \rho(e) & \text{if } 2^e - \rho(e) \leq n < 2^e. \end{cases}$$

Then $\mathrm{gd}(2^e\xi_n) \leq d_{e,n}$ *compatibly for* $n < 2^e$.

Proof. Since both composites stabilize to $2^e\xi$, the obstruction to commutativity of

$$\begin{array}{ccc} P^{2^e-\rho(e)-1} & \longrightarrow & BO_{d_{e,2^e-\rho(e)-1}} \\ \downarrow & & \downarrow \\ P^{2^e-\rho(e)} & \longrightarrow & BO_{2^e-\rho(e)} \end{array}$$

is a map $P^{2^e-\rho(e)-1} \to V_{2^e-\rho(e)}$, which is trivial for dimensional reasons. Here V_n is the fiber of $BO_n \to BO$, and is $(n-1)$-connected. The top map in this diagram comes from 2.7.(6), while the bottom map comes from 2.9. ■

Remark 2.11. If we could assert compatibility of the Eckmann liftings with those of Theorem 2.7.(6) on a larger skeleton, we might improve our immersion result to the extent mentioned in Section 1.

Remark 2.12. If one inserts the Eckmann lifting earlier in the inductive determination of $\mathrm{gd}(2^e\xi_n)$, one obtains weaker lifting results than those of 2.7.(6). For example, one can replace $g(6,7)$ by $52 = 64 - \rho(6)$, but then, by 2.10, one must also use $g(6,6) = 52$. If these values are maintained, then values of $g(7,k)$ will have to be increased for $k = 6$ and $8 \leq k \leq 14$.

Finally, in Section 6, we apply the basic induction argument, Proposition 2.5, and the results for $\mathrm{gd}(2^e\xi)$ in Proposition 2.10 to prove the following result by induction on t.

Proposition 2.13. *For $e \geq 7$ and $t \geq 1$,* $\mathrm{gd}((2^e + 2^{e+1} + \cdots + 2^{e+t})\xi_{2^e-1}) \leq 2^e - e - 7$.

This clearly implies Theorem 2.2, and hence the immersion theorem 1.1.

3. Proof of general lifting results

In this section, we prove Propositions 2.5 and 2.6. For the first one, we find it more convenient to work with sections rather than geometric dimension.

Theorem 3.1. *Let $X_0 \subset \cdots \subset X_k$ and $Y_0 \subset \cdots \subset Y_k$ be filtered spaces, and let θ (resp. η) be a vector bundle over X_k (resp. Y_k). Suppose given m_0 (resp. n_0) sections of $\theta|X_k$ (resp. $\eta|Y_k$), of which the first m_i (resp. n_i) are linearly independent (l.i.) on X_i (resp. Y_i) for $0 \le i \le k$. Let*

$$p_j = \min(m_i + n_{j-i} : 0 \le i \le j).$$

Let

$$W_j = \bigcup_{i=0}^{j} X_i \times Y_{j-i}.$$

Then there are p_0 sections of $\theta \times \eta$ on W_k of which the first p_j are linearly independent on W_j for $0 \le j \le k$. Moreover, if $\ell + i \ge j$ and $m_\ell + n_i \ge p_j$, then the first p_j sections are l.i. on $X_\ell \times Y_i$.

Note that we have $m_0 \ge \cdots \ge m_k$, $n_0 \ge \cdots \ge n_k$, and $p_0 \ge \cdots \ge p_k$.

The following result will be used in the final step of the proof of Theorem 3.1.

Lemma 3.2. *Suppose θ is an n-dimensional trivial vector bundle over a space X with l.i. sections $t_1, \ldots, t_n$. Suppose $s_1, \ldots, s_r$ are l.i. sections of θ, each of which is a linear combination with constant coefficients of the t_i. Then there is a set $s_1, \ldots, s_r, s'_{r+1}, \ldots, s'_n$ of linearly independent sections of θ.*

Proof. Because of the constant-coefficient assumption, this is just a consequence of the result for vector spaces, that a basis for a subspace can be extended to a basis for the whole space. ■

Note that the assumption about constant coefficients was required. For example, the section $s(x) = (x, x)$ of $S^2 \times \mathbb{R}^3$ cannot be extended to a set of three l.i. sections.

Proof of Theorem 3.1. Let $r_1, \ldots, r_{m_0}$ be the given sections of $\theta|X_k$, and $s_1, \ldots, s_{n_0}$ the given sections of $\eta|Y_k$. These are considered as sections of $\theta \times \eta$ by using 0 on the other component. Clearly $\{r_1, \ldots, r_{m_0}, s_1, \ldots, s_{n_0}\}$ is a set of p_0 sections on W_k which is linearly independent on W_0. The proof will proceed by finding p_1 linear combinations, always with constant coefficients, of these sections which are l.i. on

W_1, then p_2 linear combinations of these new sections which are l.i. on W_2, etc., until going into the last stage we have p_{k-1} sections which are l.i. on W_{k-1}, and we find p_k linear combinations of them which are l.i. on W_k. Now we apply the lemma repeatedly, starting with the last p_k sections. At the first step, we extend this set to a set of p_{k-1} sections l.i. on W_{k-1}, and continue until going into the last stage we have p_1 sections which are combinations of the original p_0 sections and satisfy the conclusion of the theorem for $1 \le i \le k$. We apply the lemma one last time to extend the set of p_1 sections to the desired set of p_0 sections.

Here is an explicit algorithm for the sections described in the first half of the preceding paragraph. We may assume without loss of generality that $m_0 \ge n_0$.

For j from 0 to k,

- For i from 1 to $p_j - n_0$ (resp. $p_j - m_0$), let $r_i^{(j)} = r_i$ (resp. $s_i^{(j)} = s_i$). (Note that if $n_0 \ge p_j$, then nothing happens at this step.)
- For i from $\max(1, p_j - n_0 + 1)$ to $\min(m_0, p_j)$, let both $r_i^{(j)}$ and $s_{p_j+1-i}^{(j)}$ equal $r_i^{(j-1)} + s_{p_j+1-i}^{(j-1)}$.
- Then the sections $r_i^{(j)}$ and $s_i^{(j)}$ constructed in the two previous steps give the sections which are l.i. on W_j. (Each section constructed in the second step can be counted as an r or an s, but is only counted once.)

We must show that these have the required linear independence. Before doing so, we illustrate with an example, computed by `Maple`. Let $k = 4$, $[m_0, \ldots, m_4] = [11, 6, 4, 1, 0]$ and $[n_0, \ldots, n_4] = [10, 8, 3, 2, 0]$. Then $[p_0, \ldots, p_4] = [21, 16, 14, 9, 7]$. The 16 sections l.i. on W_1 are

$$r_1, \ldots, r_6,\ r_7 + s_{10},\ r_8 + s_9,\ r_9 + s_8,\ r_{10} + s_7,\ r_{11} + s_6,\ s_5, \ldots, s_1.$$

The 14 sections l.i. on W_2 are

$$r_1,\ r_2,\ r_3,\ r_4,\ r_5 + r_7 + s_{10},\ r_6 + r_8 + s_9,\ r_7 + r_9 + s_{10} + s_8,\ r_8 + r_{10} + s_9 + s_7,$$
$$r_9 + r_{11} + s_8 + s_6,\ r_{10} + s_7 + s_5,\ r_{11} + s_6 + s_4,\ s_3,\ s_2,\ s_1.$$

The 9 sections l.i. on W_3 are

$$r_1+r_6+r_8+s_9,\ r_2+r_7+r_9+s_{10}+s_8,\ r_3+r_8+r_{10}+s_9+s_7,$$
$$r_4+r_9+r_{11}+s_8+s_6,\ r_5+r_7+r_{10}+s_{10}+s_7+s_5,\ r_6+r_8+r_{11}+s_9+s_6+s_4,$$
$$r_7+r_9+s_{10}+s_8+s_3,\ r_8+r_{10}+s_9+s_7+s_2,\ r_9+r_{11}+s_8+s_6+s_1.$$

The 7 sections l.i. on W_4 are

$$r_1+r_3+r_6+2r_8+r_{10}+2s_9+s_7,$$
$$r_2+r_4+r_7+2r_9+r_{11}+s_{10}+2s_8+s_6,$$
$$r_3+r_5+r_7+r_8+2r_{10}+s_{10}+s_9+2s_7+s_5,$$
$$r_4+r_6+r_8+r_9+2r_{11}+s_9+s_8+2s_6+s_4,$$
$$r_5+2r_7+r_9+r_{10}+2s_{10}+s_8+s_7+s_5+s_3,$$
$$r_6+2r_8+r_{10}+r_{11}+2s_9+s_7+s_6+s_4+s_2,$$
$$r_7+2r_9+r_{11}+s_{10}+2s_8+s_6+s_3+s_1.$$

Now we continue with the proof. The property described in the first paragraph of the proof, that the sections claimed to be l.i. on W_j are linear combinations of those on W_{j-1}, is clear from their inductive definition.

Next we easily show that if $i>p_j-n_0$, then

$$r_i^{(j)}=s_{p_j+1-i}^{(j)}=r_i+\sum_{\ell>i}c_\ell r_\ell+s_{p_j+1-i}+\sum_{\ell>p_j+1-i}d_\ell s_\ell$$

with c_ℓ and d_ℓ integers. The point here is that the additional terms have subscript greater than i or p_j+1-i. The proof is immediate from the inductive formula

$$r_i^{(j)}=r_i^{(j-1)}+s_{p_j+1-i}^{(j-1)}$$

and the fact that $p_j\le p_{j-1}$. Indeed, from $r_i^{(j-1)}$ we obtain terms $r_{\ge i}$ and $s_{\ge p_{j-1}+1-i}$, and from $s_{p_j+1-i}^{(j-1)}$ we obtain terms $s_{\ge p_j+1-i}$ and $r_{\ge p_{j-1}-p_j+i}$.

Finally we show that the asserted sections are l.i. on W_j. Let $\mathbf{x}\in X_\ell\times Y_{j-\ell}$. Note that $\{r_1(\mathbf{x}),\ldots,r_{m_\ell}(\mathbf{x})\}$ is l.i., as is $\{s_1(\mathbf{x}),\ldots,s_{n_{j-\ell}}(\mathbf{x})\}$, and that $p_j\le m_\ell+n_{j-\ell}$. If we form a matrix with columns labeled

$$r_1,\ldots,r_{m_0},s_{n_0},\ldots,s_1,$$

and rows which express the sections, ordered as

$$r_1^{(j)}, \dots, r_{\min(m_0,p_j)}^{(j)}, s_{p_j-m_0}^{(j)}, \dots, s_1^{(j)}, \tag{3.3}$$

in terms of the column labels, then, by the previous paragraph, the number of columns is $\geq$ (usually strictly greater than) the number of rows, the entry in position (i,i) is 1 for $i \leq \min(m_0, p_j)$, and all entries to the left of these 1's are 0. If $i > \min(m_0, p_j)$, then all entries in the r-portion of row i are 0. Moreover an analogous statement is true if the order of the rows and of the columns are both reversed. Thus there are 1's on the diagonal running up from the lower right corner of the original matrix (for $\min(n_0, p_j)$ positions) and 0's to their right.

If a linear combination of our sections applied to $\mathbf{x}$ is 0, then the triangular form of the matrix implies that the first m_ℓ coefficients are 0, while the triangular form looking up from the lower right corner implies that the last $n_{j-\ell}$ coefficients are 0. Since $p_j \leq m_\ell + n_{j-\ell}$, this implies that all coefficients are 0, hence the desired independence.

The same argument works for the last statement of the proposition. For k satisfying $j \leq k \leq \ell + i$, replace W_k by $W_k \cup (X_\ell \times Y_i)$ Then everything goes through as above. ■

Proof of Proposition 2.5. Let $D = \dim(\theta)$ and $D' = \dim(\eta)$. Then d_i, d_i', e_i, and $(X \times Y)_i$ of Proposition 2.5 correspond to $D - m_i$, $D' - n_i$, $D + D' - p_i$, and W_i of Theorem 3.1, respectively. The compatible gd bounds may be interpreted as vector bundles θ_i over X_i of dimension d_i and isomorphisms $\theta|X_i \approx \theta_i \oplus (D - d_i)$ and $\theta_i|X_{i-1} \approx \theta_{i-1} \oplus (d_i - d_{i-1})$. The trivial subbundles yield, for all i, $D - d_i$ l.i. sections of θ on X_i such that the restrictions of the sections on X_i to X_{i-1} are a subset of the sections on X_{i-1}. Each of the sections on X_0 has a largest X_i for which it is one of the given l.i. sections. By [1, 1.4.1], this section on X_i can be extended over X_k (although probably not as part of a linearly independent set). Analogous statements are true for sections of $\eta|Y_i$.

By Theorem 3.1, there are $D + D' - e_0$ l.i. sections of $\theta \times \eta$ on W_0 of which the first $D + D' - e_i$ are l.i. on W_i. Taking orthogonal complements of the spans of the sections yields the desired compatible bundles on W_i of dimension e_i, yielding the first part of Proposition 2.5.

For the second part, first note that in the algorithm in the proof of Theorem 3.1, if the r's and s's are equal, then the set of sections constructed on each W_i is invariant under the interchange map T. Thus the same will be true of the orthogonal complement of their span. ■

Proof of Proposition 2.6. (1) Let $F_1 = S^{2m-1}$ denote the fiber of $BO_{2m-1}[\rho] \to BO_{2m}[\rho]$. There is a relative Serre spectral sequence for

$$(CF_1, F_1) \to (BO_{2m}[\rho], BO_{2m-1}[\rho]) \to BO_{2m}[\rho]. \qquad (3.4)$$

The fibration $V_{2m} \to BO_{2m}[\rho] \to BO[\rho]$ shows that the bottom class of $BO_{2m}[\rho]$ is in dimension $\min(\rho, 2m)$. The spectral sequence of (3.4) shows that $H_*(S^{2m}) \to H_*(BO_{2m}[\rho]/BO_{2m-1}[\rho])$ has cokernel beginning in dimension $s+1$, and so the map is an s-equivalence. Thus the inclusion of the s-skeleton of $BO_{2m}[\rho]/BO_{2m-1}[\rho]$ factors through S^{2m} to yield the map c_1', which is an equivalence.

The second map is obtained similarly. A map $\Sigma P^{2m}_{2m-1} \xrightarrow{g} BO_{2m+1}[\rho]/BO_{2m-1}[\rho]$ is obtained as the inclusion of a skeleton of CF_2/F_2, where $F_2 = V_{2m+1,2}$ is the fiber of $BO_{2m-1}[\rho] \to BO_{2m+1}[\rho]$. The relative Serre spectral sequence of

$$(CF_2, F_2) \to (BO_{2m+1}[\rho], BO_{2m-1}[\rho]) \to BO_{2m+1}[\rho] \qquad (3.5)$$

implies that $\operatorname{coker}(g_*)$ begins in dimension $s+1$, determined by $H_{2m}(CF_2, F_2) \otimes H_{\min(\rho,2m+1)}(BO_{2m+1}[\rho])$ and the first "product" class in $H_{4m}(\Sigma V_{2m+1,2})$. The obtaining of c_2' now follows exactly as for c_1'.

(2) Let $Q := BO_{2m+1}[\rho]/BO_{2m-1}[\rho]$ and $E := \text{fiber}(BO_{2m+1}[\rho] \to Q)$. The commutative diagram of fibrations

$$\begin{array}{ccccc}
V_{2m+1,2} & \longrightarrow & BO_{2m-1}[\rho] & \longrightarrow & BO_{2m+1}[\rho] \\
\downarrow & & \downarrow & & \downarrow \\
\Omega Q & \longrightarrow & E & \longrightarrow & BO_{2m+1}[\rho]
\end{array}$$

implies that the quotient $E/BO_{2m-1}[\rho]$ has the same connectivity as $\Omega Q/V_{2m+1,2}$, which is 1 less than that determined from (3.5); that is, $E/BO_{2m-1}[\rho]$ is $(s-1)$-connected. Thus, since $\dim(X) < s$, the vertical maps in

$$\begin{tikzcd} BO_{2m-1}[\rho] \arrow[r] \arrow[d] & BO_{2m+1}[\rho] \arrow[r] \arrow[d] & \Sigma P^{2m}_{2m-1} \arrow[d] \\ E \arrow[r] & BO_{2m+1}[\rho] \arrow[r] & Q \end{tikzcd}$$

are equivalences in the range relevant for maps from X, A, and X/A. Since the bottom row is a fibration, we may consider the top row to be one, too, as far as X is concerned.

Since g is divisible by 2, and $2\pi_{2m}(\Sigma P^{2m}_{2m-1}) = 0$, we deduce that the composite

$$X/A \xrightarrow{g} S^{2m} \xrightarrow{i} \Sigma P^{2m}_{2m-1}$$

represents the 0 element of $[X/A, \Sigma P^{2m}_{2m-1}]$; i.e. the map is null-homotopic rel $*$. There is a commutative diagram as below with the left sequence a cofiber sequence and the right sequence a fiber sequence in the range of $\dim(X)$.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "j_1"'] & BO_{2m-1}[\rho] \arrow[d, "j_2"'] \\ X \arrow[r, "f"] \arrow[d, "q"'] & BO_{2m+1}[\rho] \arrow[d] \\ X/A \arrow[r, "i\circ g"] & \Sigma P^{2m}_{2m-1} \end{tikzcd} \tag{3.6}$$

We have just seen that there is a basepoint-preserving homotopy

$$H : X/A \times I \to \Sigma P^{2m}_{2m-1}$$

from $i \circ g$ to a constant map. There is a commutative diagram

$$\begin{tikzcd} X \times 0 \cup A \times I \arrow[rr] \arrow[d] & & BO_{2m+1}[\rho] \arrow[d] \\ X \times I \arrow[r, "q\times I"] & X/A \times I \arrow[r, "H"] & \Sigma P^{2m}_{2m-1} \end{tikzcd}$$

where the top map is f on $X \times 0$ and $j_2 \circ f_1$ on each $A \times \{t\}$. By the Relative Homotopy Lifting Property of a fibration, there exists a map $\widetilde{H} : X \times I \to BO_{2m+1}[\rho]$ making both triangles commute. When $t=1$, it maps into $BO_{2m-1}[\rho]$, since it projects to the constant map at the basepoint of ΣP_{2m-1}^{2m}.

(3) We use the fact that $2\cdot 1_{\Sigma P_{2m-1}^{2m}}$ factors as

$$\Sigma P_{2m-1}^{2m} \xrightarrow{\text{col}} S^{2m+1} \xrightarrow{\eta} S^{2m} \hookrightarrow \Sigma P_{2m-1}^{2m}$$

to deduce that the composite

$$X/A \xrightarrow{g'} \Sigma P_{2m-1}^{2m} \xrightarrow{\text{col}} S^{2m+1}$$

is null-homotopic since g' is divisible by 2. An argument similar to the one in the beginning of the proof of (2) shows that $BO_{2m}[\rho]\to BO_{2m+1}[\rho]\to S^{2m+1}$ is a fibration through dimension $\min(\rho+2m,4m+1)\ge s+1$. Since $\dim(X)\le s+1$, the lifting follows as in the proof of (2). However, we need $\dim(X)\le s$ because the map c_2 in (1) only exists on the s-skeleton. ■

4. Inductive determination of a bound for $\mathrm{gd}(2^e,k)$

In this section, we prove that part (6) of Theorem 2.7 follows from its first five parts, together with initial values of $g(e,k)$ given in Table 2.8 when $k=1$ or $e=4$.

We begin by noting that 2.7.(6) is true for $e=4$, since, by [7, 6.1] or Proposition 2.9, $\mathrm{gd}(16\xi_{15})\le 7$. The compatibility requirement is trivially satisfied because there are only three values for the number of sections involved—no sections, full sections (i.e. trivial bundle), and one intermediate value. Indeed $16\xi_n$ has no sections for $n\ge 16$, 16 sections for $n\le 8$, and at least (and in fact exactly) 9 sections for $8\le n\le 15$. These values, $\mathrm{gd}(16,0)=0$, $\mathrm{gd}(16,1)=7$, and $\mathrm{gd}(16,2)=16$ agree with the values of $g(e,k)$ tabulated in Table 2.8.

Let $\rho=\rho[e-1]$. Assume that we have obtained compatible liftings of $2^{e-1}\xi_{8k+7}$ to $BO_{g(e-1,k)}[\rho]$ for all k. For $0\le k\le 2^{e-3}$, define

$$g_1(e,k) := \max\{g(e-1,i)+g(e-1,k-i) : \max(0,k-2^{e-4})\le i\le [k/2]\}.$$

Note that by 2.7.(3),

$$g(e,k)\ge g_1(e,k)-1. \tag{4.1}$$

Recall $A_k = P^{8k+7}$, and let

$$(A \times A)_k = \bigcup_{i=0}^{k} A_i \times A_{k-i}.$$

Then by Proposition 2.5 there are compatible symmetric liftings ℓ_k of $2^{e-1}\xi \times 2^{e-1}\xi$ on $(A\times A)_k$ to $BO_{g_1(e,k)}[\rho]$ for all k. We precede by compatible maps $d_k : A_k \to (A\times A)_k$, cellular maps homotopic to the diagonal. The composites $A_k \xrightarrow{\ell_k \circ d_k} BO_{g_1(e,k)}[\rho]$ are compatible liftings of $2^e\xi_{8k+7}$ for all k.

By decreasing induction on k starting with $k = 2^{e-3}$, we will construct compatible factorizations through $BO_{g(e,k)}[\rho]$ of the maps $\ell_k \circ d_k$. Assume inductively that, for all $j > k$, compatible factorizations, up to homotopy rel A_k, of $\ell_j \circ d_j$ through $BO_{g(e,j)}[\rho]$ have been attained. If $g(e,k) \ge g_1(e,k)$, then no factorization of $\ell_k \circ d_k$ is required, and so our induction on k is extended. So we may assume $g(e,k) = g_1(e,k) - 1$.

Let $h = [k/2]$. Let k' be the largest integer less than k such $g(e,k') < g(e,k)$. By 2.7.(5), $g(e,k') < g(e,k) - 1$, and hence by (4.1)

$$g_1(e,k') \le g(e,k) - 1. \tag{4.2}$$

By (4.2), 2.7.(4), and the last part of Proposition 3.1 (which is required for compatibility of the lifts of $(A\times A)_{k'}$ and $A_h \times A_h$ to $BO_{g(e,k)-1}$), we have the commutative diagram below, similar to (3.6).

$$\begin{tikzcd}
A_{k'} \arrow[r, "d'"] \arrow[dd] & (A\times A)_{k'} \cup A_h \times A_h \arrow[r] \arrow[dd] & BO_{g(e,k)-1}[\rho]^{(8k+7)} \arrow[d] \\
 & & BO_{g(e,k)}[\rho]^{(8k+7)} \arrow[d] \\
A_k \arrow[r, "d_k"] \arrow[d] & (A\times A)_k \arrow[r, "\ell_k"] \arrow[d] & BO_{g(e,k)+1}[\rho]^{(8k+7)} \arrow[d, "c"] \\
A_k/A_{k'} \arrow[r, "\overline{d}"] & (A\times A)_k/((A\times A)_{k'} \cup A_h \times A_h) \arrow[r, "\overline{\ell}"] & C,
\end{tikzcd}$$

where $C = S^{g(e,k)+1}$ if $g(e,k)$ is odd, and $C = \Sigma P^{g(e,k)}_{g(e,k)-1}$ if $g(e,k)$ is even. The maps labeled d are cellular maps homotopic to the diagonal. The map c is obtained similarly to the first paragraph of the proof of 2.6, using 2.7.(2)[2] to conclude that $g(e,k) \geq 4k+4$, provided $k>1$. We will deal with the case $k=1$ at the end of this proof.

The quotient $(A\times A)_k/(A_h\times A_h)$ equals $B\vee T(B)$, where T reverses the order of the factors, and B is the union of all cells $e^i\times e^j$ with $i<j$. By the symmetry property of ℓ_k, $\overline{\ell}|T(B)=(\overline{\ell}|B)\circ T$. Since $T\circ\overline{d}\simeq\overline{d}$, we conclude that $\overline{\ell}\circ\overline{d}$ is divisible by 2. Indeed, with r_B denoting the retraction onto B,

$$[\overline{\ell}\circ\overline{d}]=[(\overline{\ell}|B)\circ r_B\circ\overline{d}]+[(\overline{\ell}|T(B))\circ r_{T(B)}\circ\overline{d}]$$

and we have

$$[(\overline{\ell}|T(B))\circ r_{T(B)}\circ\overline{d}]=[(\overline{\ell}|T(B))\circ T\circ r_B\circ\overline{d}]=[(\overline{\ell}|B)\circ r_B\circ\overline{d}].$$

Thus, by Proposition 2.6, $\ell_k\circ d_k$ is homotopic rel $A_{k'}$ to a map which lifts to $BO_{g(e,k)}[\rho]$. Note that the lifting into $BO_{g(e,k)-1}[\rho]$ was not needed if $g(e,k)$ is odd. Hence, once we handle the case $k=1$ postponed above, we will have extended our inductive lifting hypothesis, and so will have proved that there are compatible liftings of A_k to $BO_{g(e,k)}[\rho]$ for all k. This extends the induction on e and proves Theorem 2.7.(6), assuming the first five parts of 2.7.

The case $k=1$ was postponed above. We consider it here. The subtlety is that we are asserting a lifting outside the stable range. We consider primarily the case $e=5$. The case $e=4$, discussed at the beginning of the section, has yielded a commutative diagram

$$\begin{array}{ccc}
P^7 & \longrightarrow & BO_0[9] \\
\downarrow & & \downarrow \\
P^{15} & \longrightarrow & BO_7[9] \\
\downarrow & & \downarrow \\
P^{23} & \longrightarrow & BO_{16}[9].
\end{array}$$

The maps factor through maps on P_9^{23}, P_9^{15}, and $P_9^7=*$. The map $P_9^{15}\xrightarrow{f}BO_7[9]$ is obtained (see [8, 3.2]) as a compression of

$$P_8^{15}\xrightarrow{r}S^8\xrightarrow{2g}BO[9],$$

[2]By 2.7.(2), if $g(e,k)<4k+4$, then $e>4k+2$, and hence $\mathrm{gd}(2^e\xi_{8k+7})=0$, in which case we are done.

which lifts to $BO_7[9]$ by [3, 2.1]. Note that we could have chosen any lifting of the stable map 16ξ to BO_7, and we chose this one. By [3, 2.1], $[2f]$ lifts to BO_6, and $[4f]$ to BO_5.

Proposition 2.5 yields a commutative diagram

$$\begin{tikzcd}
P_9^{15} \arrow[r, "d_1"] \arrow[d] & (P_9^{15} \times P_9^{15})^{(15)} \arrow[r] \arrow[d] & BO_7[9] \arrow[d] \\
P_9^{23} \arrow[r, "d_2"] \arrow[d] & (P_9^{23} \times P_9^{23})^{(23)} \arrow[r] \arrow[d] & BO_{16}[9] \arrow[d] \\
P_9^{31} \arrow[r, "d_3"] \arrow[d] & (P_9^{31} \times P_9^{31})^{(31)} \arrow[r] \arrow[d] & BO_{23}[9] \arrow[d] \\
P_9^{39} \arrow[r, "d_4"] & (P_9^{39} \times P_9^{39})^{(39)} \arrow[r] & BO_{32}[9],
\end{tikzcd}$$

where the horizontal maps are liftings of $16\xi \times 16\xi$, and d_i are cellular maps homotopic to the diagonal. Proposition 2.6 then allows an improvement to a commutative diagram

$$\begin{tikzcd}
P_9^{15} \arrow[r] \arrow[d] & P_9^{15} \vee P_9^{15} \arrow[r] & BO_7[9] \arrow[d] \\
P_9^{23} \arrow[rr] \arrow[d] & & BO_{15}[9] \arrow[d] \\
P_9^{31} \arrow[rr] \arrow[d] & & BO_{22}[9] \arrow[d] \\
P_9^{39} \arrow[rr] & & BO_{32}[9].
\end{tikzcd}$$

We could not use 2.6 to lift the top map to BO_6 because the dimensional conditions were not satisfied. However, the class of this map is 2 times the map f described in the preceding paragraph, and hence, by the argument there, it lifts to BO_6, as desired. A very similar argument works when $e = 6$ to lift to BO_5.

5. The function $g(e,k)$

In this section, we define the function $g(e,k)$ which has been used in the previous sections, and prove the first five parts of Theorem 2.7, its numerical properties which were already used to prove 2.7.(6), its important geometrical property.

Let $\mathrm{lg}(k) = [\log_2(k)]$. Except for an irregularity when $k = 1$, g is determined by $g(k, \mathrm{lg}(k) + 4)$, which we will denote by $f(k)$. For $k \le 16$, this function $f(k)$ of one variable has the values indicated in boldface in Table 2.8. The companion equations relating f and g are

$$f(k) = g(\mathrm{lg}(k) + 4, k), \tag{5.1}$$

and, if $k > 1$,

$$g(e,k) = \begin{cases} 2^e & e \le \mathrm{lg}(k) + 3 \\ \max(4k+4, f(k) - e + \mathrm{lg}(k) + 4) & \mathrm{lg}(k) + 4 \le e \le 4k+2 \\ 0 & e > 4k+2. \end{cases} \tag{5.2}$$

For $k = 1$, the values of g are as in Table 2.8. The reason for the irregularity when $k = 1$ is that in the previous section we used special considerations to get liftings of bundles over P^{15} beyond the stable range. It was important to do this to get a good start on the induction.

The formula for f is too complicated to write explicitly, largely due to requirement 2.7.(5). It utilizes, among other things, the following auxiliary function.

Definition 5.3.

$$\delta(n) = \max(\nu(n) - 1, \min(2, \delta'(n))),$$

where

$$\delta'(n) = \max\{\nu(n-d) - 4d + 3 :\ 2 \le d < n\}.$$

Recall that $\nu(-)$ denotes the exponent of 2. For example, $\delta(n) = \nu(n) - 1$ if $n \equiv 0$ mod 8, while

$$\delta(n) = \begin{cases} 1 & \text{if } \nu(n-2) = 6 \\ 2 & \text{if } \nu(n-2) \ge 7 \\ 0 & \text{if } \nu(n-3) = 9 \end{cases}$$

are the only cases with $n < 3^{10} + 3$ for which $\delta(n) \ne \nu(n) - 1$.

A first approximation to f is given by

$$f_0'(n) = 8n - \lg(n) + \delta(n).$$

We explain now the rationale behind the definition of δ. The $(\nu(n)-1)$-part is present just to make the basic induction work and to agree with some initial values. If it were not for two delicate matters, we could just define $f(n) = 8n - \lg(n) + \nu(n) - 1$. The first of these delicate matters is the stability requirement in 2.7.(2), illustrated by the last two 12's in column 2 and the last three 16's in column 3 of Table 2.8. The smallest n for which $\delta(n) \neq \nu(n) - 1$ is $n = 66$; $\delta(66) = 1$, using $d = 2$ in 5.3. The need for this is seen in

$$522+\delta(66) = 8\cdot 66-6+\delta(66) = f(66) = g(10,66) \geq g(9,64)+g(9,2)-1 = 512+12-1.$$

Note from Table 2.8 that the $12 = g(9,2)$ is 1 greater than it would have been were it not for the stability considerations. This "1" is intimately related to $\delta(66) = 1$.

The other delicate matter is that 2.7.(5) requires $f(n) \neq f(n-1)+1$. This is the cause of most of the complications. Recall that the reason that this is so important goes back to Proposition 2.6.(3), which requires that if you want to utilize paired obstructions to lift from $2m+1$ to $2m$, compatibly with a given lifting on a subcomplex, then that lifting must be to $2m-1$.

Now we give the definition of f.

Definition 5.4. *For $n \geq 1$, let $f_0'(n) = 8n - \lg(n) + \delta(n)$ and*

$$f_0(n) = \max\{f_0'(m) : m \leq n\}.$$

Note that f_0 is an increasing function of n. Define $s(n)$ and $f(n) := f_0(n) + s(n)$ inductively by $f(0) = 0$ and for $n \geq 1$

$$s(n) = \begin{cases} 1 & \text{if } f_0(n) = f(n-1) \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the $-$ and $+$ parts of $\pm$ are present for different reasons. The $-$ is present to make f increasing, while the $+$ occurs to prevent $f(n) - f(n-1) = 1$. For example, if $f(n-2) = f_0(n-2) = A$, $f_0(n-1) = A+1$, and $f_0(n) = A+1$, then $s(n-1) = 1$ for the latter reason, yielding $f(n-1) = A+2$, and so $s(n) = 1$ for the former reason.

Then f is an increasing function such that $f(n)-f(n-1)\neq 1$. Before proving that f (actually the associated g) satisfies the required properties of 2.7, we give a few examples.

Example 5.5. *Let u always denote an odd positive integer, and A an arbitrary positive integer.*

- *$s(n)=0$ if $n<2^7+1$, because $f_0'(n)-f_0'(n-1)\geq 8-\nu(n-1)$.*
- *$s(u\cdot 2^7+1)=1$ for most odd integers u. When $u=1$, this is due to*
$$f_0'(2^7+\epsilon)=2^{10}+\begin{cases}-15 & \epsilon=-1\\ -1 & \epsilon=0\\ 0 & \epsilon=1\\ 9 & \epsilon=2.\end{cases}$$
However, if $e\geq 2^9+1$, then $s(2^e+2^7+1)=0$ due to $\delta(2^e+2^7+1)>-1$.
- *If $k\geq 2$ and $m_k=\max(\nu(k)-1,2)$, then for most odd integers u, $s(u\cdot 2^{8k+m_k}+k)=1$. When $k=2$, this is due to*
$$f_0'(2^{18}+\epsilon)=2^{21}+\begin{cases}-26 & \epsilon=-1\\ -1 & \epsilon=0\\ -11 & \epsilon=1\\ 0 & \epsilon=2\\ 5 & \epsilon=3.\end{cases}$$
- *$s(u\cdot 2^{2^{10}+1}+2^8+1)=1$ for most u, using $d=2^8+1$ in $\delta(n)$ in 5.3. Similarly, $s(2^{2^{11}+2}+2^9+1)=1$ and $s(A\cdot 2^{2^{12}+3}+2^{10}+1)=1$.*
- *$s(2^{2^{20}+16}+2^{17}+\epsilon)=1$ for $0\leq\epsilon\leq 2$, since*
$$f_0(2^{2^{20}+16}+t)=2^{2^{20}+19}+\begin{cases}-1 & 0\leq t<2^{17}\\ 0 & 2^{17}\leq t\leq 2^{17}+1\\ 2 & t=2^{17}+2.\end{cases}$$
- *Similarly, $s(2^{2^{12}+8}+2^9+\epsilon)=1$ for $0\leq\epsilon\leq 1$.*

Now we proceed to the proof of the relevant portions of 2.7. The reader can easily check that the formulas (5.2) and 5.4 agree with Table 2.8 in its limited range.

Properties (1) and (2) of 2.7 are immediate from (5.2). To establish property (5), we first note that $f(n) \neq f(n-1)+1$ by Definition 5.4. This and (5.2) imply 2.7.(5) with the only minor worry being about the truncation of $g(e,k)$ at $4k+4$, but this only occurs when $\nu(k) < 4$, and from this it follows easily that $g(e,k+1) > g(e,k)+1$ in these cases. Observation of columns 2, 3, and 4 of Table 2.8 should convince the reader that this is true.

We now proceed to prove 2.7.(3). We first prove that it is true using f_0. First note that we need not worry that f_0 is a max, for the sum of the parameters which yield the max for the terms of the RHS of 2.7.(3) will be admissible for the evaluation of the LHS. It suffices to prove 2.7.(3) when $e = \lg(k)+4$, in which case it reduces to

$$f_0'(k) \geq f_0'(\ell) - (\lg(k)+3) + \lg(\ell) + 4 + f_0'(k-\ell) - (\lg(k)+3) + \lg(k-\ell) + 4 - 1, \tag{5.6}$$

unless $g(e-1,\ell) = 4\ell+4$ due to truncation. We will deal with this possibility later. Using 5.4, (5.6) reduces to

$$\delta(k) + \lg(k) \geq \delta(\ell) + \delta(k-\ell) + 1, \tag{5.7}$$

provided $2 \leq \ell \leq k-2$.

The case $\ell = 1$ is easily handled. Then $g(\lg(k)+3, \ell)$, which occurs in the simplification to (5.6), is 0 unless $\lg(k) \leq 3$, and 2.7.(3) is easily verified in these cases.

Returning to the proof of (5.7), write $k = 2^a + m$ with $0 \leq m < 2^a$. Cases with $a \leq 6$ are easily checked directly, and so we assume $a \geq 7$. Assume without loss of generality that $\nu(\ell) \geq \nu(k-\ell)$. If $\nu(\ell) < 3$, then (5.7) is clearly satisfied since its LHS is ≥ 6, while its RHS is ≤ 5.

Now we may assume $\nu(\ell) \geq 3$, and hence $\delta(\ell) = \nu(\ell) - 1$. If $k \equiv 0 \mod 8$, then $\nu(k) \geq \nu(k-\ell)$, and so (5.7) follows from $a \geq \nu(\ell)$. Hence we may assume $k \not\equiv 0 \mod 8$. Then $\nu(k-\ell) = \nu(k)$ and $\delta(k-\ell) - \delta(k) \leq 2-(-1) = 3$, so that the only way (5.7) might fail is if $\nu(\ell) \geq a-2$; i.e., if $\ell = 2^a$, 2^{a-1}, 2^{a-2}, $3 \cdot 2^{a-2}$, or possibly (depending on how large m is) $3 \cdot 2^{a-1}$, $5 \cdot 2^{a-2}$, and $7 \cdot 2^{a-2}$. One easily verifies that (5.7) holds in these cases. For example, if $\ell = 2^a$, then (5.7) reduces to showing $\delta(2^a+m) \geq \delta(m)$. This is true because any value of d in 5.3 that causes $\delta(m)$ to be greater than its minimal value will also cause the same for $\delta(2^a+m)$. If $\ell = 2^{a-1}$, then (5.7) reduces to $\delta(2^a+m) \geq \delta(2^{a-1}+m) - 1$, which is true with 1 to spare.

We complete our proof of the f_0-version of 2.7.(3) by considering what happens in the postponed case in which $g(e-1,\ell) = 4\ell+4$ due to truncation. The definition of the function δ has been formulated to handle this case. We begin by illustrating with the case $\ell = 3$, $e = 15$. Note that $g(14,3)$ is the lowest 16 in the $k = 3$ column of Table 2.8, and is 3 larger than it would have been if the values of $g(e,3)$ were allowed to decrease below 16. In the context of (5.7), that would add 3 to the RHS. Since $e = \lg(k)+4$ in (5.6), we have $\lg(k) = 11$. So $k = 2^{11}+t$, $0 \le t < 2^{11}$, and we need to verify

$$\delta(2^{11}+t) + 11 \ge \delta(3) + \delta(2^{11}+t-3) + 4. \tag{5.8}$$

Since $\delta(3) = -1$, (5.8) reduces to

$$\delta(2^{11}+t) + 8 \ge \delta(2^{11}+t-3). \tag{5.9}$$

The only way this could fail is if $\nu(2^{11}+t-3) \ge 9$. In Table 5.10, we tabulate the values of both sides of (5.9) for these values of t, and see that (5.9) holds in each. The definition of δ has been formulated so that it will always work this way.

Table 5.10. Verification of (5.9).

t	$\delta(2^{11}+t)+8$	$\delta(2^{11}+t-3)$
3	$2+8$	10
2^9+3	$0+8$	8
$2^{10}+3$	$1+8$	9
$3\cdot 2^9+3$	$0+8$	8

The general case of 2.7.(3) when truncation occurs is extremely similar. Let $\ell > 1$ be arbitrary.[3] The worst case occurs when $e = 4\ell+3$, because then $g(e-1,\ell)$ is the last nonzero entry in its column. The amount of truncation is $\max(2-\delta(\ell),0)$. This is achieved from (5.2) and 5.4 as

$$4\ell+4-(f_0'(\ell)-(4\ell+2)+\lg(\ell)+4) = 8\ell+2-(8\ell-\lg(\ell)+\delta(\ell))-\lg(\ell).$$

Since $e = \lg(k)+4$ in (5.6), we have $k = 2^{4\ell-1}+t$ with $0 \le t < 2^{4\ell-1}$. The analogue of (5.7), which we must establish, is

$$\delta(2^{4\ell-1}+t)+4\ell-1 \ge \delta(\ell)+\delta(2^{4\ell-1}+t-\ell)+1+(2-\delta(\ell)),$$

[3]There is no truncation when $\ell = 1$.

which reduces to

$$\delta(2^{4\ell-1}+t)+4\ell \geq \delta(2^{4\ell-1}+t-\ell)+4. \tag{5.11}$$

This inequality is easily verified, using the definition of δ, as follows:

$$\begin{aligned} \text{LHS} \geq 7 \geq \text{RHS} &\quad \text{if} \quad \nu(t-\ell) \leq 3 \\ \text{LHS} \geq 4\ell-1 \geq \text{RHS} &\quad \text{if} \quad 3 \leq \nu(t-\ell) < 4\ell-3 \\ \text{LHS} \geq \nu(t-\ell)+3 \geq \text{RHS} &\quad \text{if} \quad 4\ell-3 \leq \nu(t-\ell) \leq 4\ell-1. \end{aligned}$$

Note that $\nu(t-\ell)$ can be no larger than $4\ell-1$. The first inequality in the third line follows from 5.3.

Having now verified 2.7.(3) when f_0 is used, we next show that it follows that this is also valid when f is used. Because $0 \leq f - f_0 \leq 1$, the principal worry is to show that if equality was attained in (5.7) or (5.11) using f_0, then it cannot happen that $s = 1$ on the RHS but not on the LHS, causing the inequality to fail.[4] Equality occurs in (5.7) using f_0 only when $k = 2^a + m$, $0 \leq m < 2^a$, and $\ell = 2^a$. Thus we need to show here that

$$(\delta+s)(2^a+m) \geq (\delta+s)(m). \tag{5.12}$$

Some typical occurrences of $s(n) = 1$ were given in Example 5.5 and a complete description of these is given in Lemma 6.7. It follows from this that the only way that we can have $s(m) = 1$ while $s(2^a + m) = 0$ is if $\delta(2^a + m) > \delta(m)$, as occurred for $m = 2^7 + 1$ and $a \geq 2^9 + 1$ in the second bullet in 5.5. In such cases, (5.12) is necessarily satisfied because of the increase in δ. In the notation of Lemma 6.7, if $m = A_0 + \cdots + A_t$ has $s(m) = 1$, then $2^a + m$ with $2^a > m$ can be written with A_0 replaced by $2^a + A_0$, with $\nu(-)$ unchanged. Then the condition which caused $s(m) = 1$ will also cause $s(2^a + m) = 1$ unless $\delta(2^a + m) \neq \delta(m)$. However, adding a large 2-power such as 2^a cannot decrease δ. Thus $\delta(2^a + m) > \delta(m)$ and hence (5.12) is satisfied.

The only cases of equality in (5.11) occur when $\nu(t-\ell) \geq 4\ell-4$. Thus the $(\delta+s)$-version of (5.11) could fail only if $s(2^{4\ell-1}+u2^e) = 1$ with $e \geq 4\ell-4$ (and $u2^e < 2^{4\ell-1}$). But Lemma 6.7 shows that $s(n) = 1$ only when n has at least one long string of 0's

[4] The possibility of two cases of $s = 1$ on the RHS of (5.7) when the inequality was satisfied with 1 to spare can be eliminated similarly.

in its binary expansion, which is not the case for $n = 2^{4\ell-1} + u2^e$ with $e \le 4\ell - 4$ and $u2^e < 2^{4\ell-1}$. This completes the proof of 2.7.(3).

Next we prove 2.7.(4). Note that it is similar to 2.7.(3), except it is stronger by 2. Note also from Table 2.8 that the claim is false when $2\ell = 2^{e-3}$, for we have $g(e, 2^{e-3}) = 2^e = 2g(e-1, 2^{e-4})$. The exclusion on the other side of 2.7.(3), when $[(e+1)/4] > 2\ell$, is because both $g(e, 2\ell) = 0$ and $g(e-1, \ell) = 0$ in this case. Similarly to (5.7), the claim reduces to

$$(\delta + s)(2\ell) + \lg(2\ell) \ge 2(\delta + s)(\ell) + 3. \tag{5.13}$$

Note that for $\ell < 2^8$,

$$(\delta + s)(\ell) = \begin{cases} \nu(\ell) & \text{if } \ell = 2^6 + 2,\ 2^7 + 1, \text{ or } 2^7 + 2 \\ \nu(\ell) - 1 & \text{otherwise.} \end{cases}$$

For these three special values of ℓ, (5.13) is easily verified, while if $(\delta+s)(\ell) = \nu(\ell)-1$, then (5.13) reduces to $\lg(\ell) \ge \nu(\ell)$, which is clearly true. Thus (5.13) is true for $\ell < 2^8$. The reason that this analysis didn't catch the failure of 2.7.(4) to hold when $2\ell = 2^{e-3}$ is that the analysis deals with $f(-)$, and the values $g(e, 2^{e-3}) = 2^e$ sit above the f-values in Table 2.8; for example, $f(8) = g(7, 8) = 63$, not the 64 which sits above it in the table.

If $\ell \ge 2^8$, so that $\lg(2\ell) \ge 9$, and $\delta(\ell) < 3$, then (5.13) is certainly true. Here we use 5.5 or 6.7 to see that s cannot play a significant role here; the second value of n with $s(n) > 0$ is $s(2^{18}+2) = 1$, for which $\lg(n)$ will certainly make (5.13) hold. Thus, we may assume $\ell \equiv 0 \mod 8$ and then $\delta(\ell) = \nu(\ell) - 1$, and hence (5.13) reduces to

$$\lg(\ell) + s(2\ell) \ge \nu(\ell) + 2s(\ell). \tag{5.14}$$

Note that

$$\lg(\ell) - \nu(\ell) = \begin{cases} 0 & \ell = 2^a \\ 1 & \ell = 2^{a-1} \\ \ge 2 & \text{otherwise.} \end{cases}$$

Since $s(2^e) = 0$, we deduce that (5.14) holds.

6. Inductive determination of a bound for gd of normal bundle

In this section, we prove the following result, of which Proposition 2.13 is an immediate consequence.

Theorem 6.1. *Let $e \ge 7$ and*

$$d'_{e,n} = \begin{cases} 0 & \text{if } n \le \rho(e) \\ g(e, [\frac{n}{8}]) & \text{if } \rho(e) < n < 2^e - \rho(e) \\ \max(g(e, [\frac{n}{8}]) - 1, 2^e - \rho(e)) & \text{if } 2^e - \rho(e) \le n \le 2^e - 9 \\ 2^e - e - 7 & \text{if } 2^e - 8 \le n \le 2^e - 1. \end{cases}$$

For $t \ge 1$, $\mathrm{gd}((2^e + 2^{e+1} + \cdots + 2^{e+t})\xi_n) \le d'_{e,n}$ compatibly for $n < 2^e$.

Remark 6.2. The all-important $2^e - e - 7$ arises from the bound $g(e, 2^{e-4} - 1) + g(e+1, 2^{e-4}) = 2^{e-1} - e - 5 + 2^{e-1} - 2$ for $\mathrm{gd}((2^e + 2^{e+1})\xi_{2^e-1})$.

Example 6.3. We illustrate the argument when $e = 7$. Here we have

$$d'_{e,n} = \begin{cases} 100 & \text{if } 104 \le n \le 111 \\ 112 & \text{if } 112 \le n \le 119 \\ 114 & \text{if } 120 \le n \le 127. \end{cases}$$

By Proposition 2.10, we can replace the 109 and 116 in the ($e = 7$)-row of Table 2.8 by 112 and 112. Call the values in this modified row $g'(7, k)$. These are compatible bounds for $\mathrm{gd}(2^7, k)$. Apply Propositions 2.5 and 2.6 to this to get a modified ($e = 8$)-row, for $k \le 15$, with the 108 and 115 replaced by 111 and 114. Call the values in this new row $g'(8, k)$. The 111 for $g'(8, 14)$ is determined by $g'(7, 14) + g'(7, 0) - 1 = 112 + 0 - 1$, while the 114 for $g'(8, 15)$ is determined by $g'(7, 7) + g'(7, 8) - 1 = 52 + 63 - 1$. Now apply Proposition 2.5 to $g'(7, k)$ and $g'(8, k)$ to obtain compatible bounds for $\mathrm{gd}((2^7 + 2^8)\xi_n)$, $n \le 127$. The value $d'_{7,119} = 112$ is determined by $g'(7, 14) + g'(8, 0) = 112 + 0$, while $d'_{7,127} = 114$ is determined by $g'(7, 0) + g'(8, 15) = 0 + 114$ or $g'(7, 7) + g'(8, 8) = 52 + 62$. Applying Proposition 2.5 to the $d'_{7,n}$ bounds for $\mathrm{gd}((2^7 + 2^8)\xi_n)$ and the $g(9, k)$ bounds for $\mathrm{gd}(2^9\xi_{8k+7})$ maintains the $d'_{7,n}$ bound for $\mathrm{gd}((2^7 + 2^8 + 2^9)\xi_n)$, and the addition of larger $2^e\xi$ is handled in the same way.

Proof of Theorem 6.1. The above example when $e = 7$ was slightly simpler than the general situation because $\rho(7) \equiv 0 \bmod 8$. Each value $g(e, k)$ gives a bound for $\mathrm{gd}(2^e\xi_i)$ for $8k \le i \le 8k + 7$. If $\rho(e) \not\equiv 0 \bmod 8$, then the skip of $d'_{e,n}$ at $n = 2^e - \rho(e)$ occurs in the middle of one of these ranges, forcing a refinement of the filtering of P^{2^e-1}. It becomes convenient to filter it using all skeleta P^i.

The proof will proceed in five steps.

(1) Use Proposition 2.10 for $\mathrm{gd}(2^e\xi_n)$ for $n < 2^e$.

(2) Use (1) and Propositions 2.5 and 2.6 to prove

$$\mathrm{gd}(2^{e+1}\xi_n) \le \begin{cases} 0 & n \le \rho(e+1) \\ g(e+1,[\frac{n}{8}]) & \rho(e+1) < n < 2^e - \rho(e) \\ \max(g(e+1,[\frac{n}{8}]), 2^e - \rho(e) - 1) & 2^e - \rho(e) \le n \le 2^e - 9 \\ 2^e - e - 7 & 2^e - 8 \le n \le 2^e - 1 \end{cases}$$

compatibly for $n < 2^e$.

(3) Use (1) and (2) and Proposition 2.5 to prove

$$\mathrm{gd}((2^e + 2^{e+1})\xi_n) \le d'_{e,n}$$

compatibly for $n < 2^e$.

(4) By induction on t, using (2) to get started and Propositions 2.5 and 2.6, show $\mathrm{gd}(2^{e+t}\xi_n)$ has the same bound as in (2), compatibly for $n < 2^e$. We can actually do better than this, but this is all we need.

(5) By induction on t, using (3) to get started and then also (4) and 2.5, show that $\mathrm{gd}((2^e + 2^{e+1} + \cdots + 2^{e+t})\xi_n) \le d'_{e,n}$ compatibly for $n < 2^e$, completing the proof of the theorem.

Step (1) is immediate, and steps (4) and (5) are similar to and easier than steps (2) and (3), respectively. We now prove step (2).

For $n < 2^e - \rho(e)$, this is Theorem 2.7.(6), which has already been proven. For $2^e - \rho(e) \le n \le 2^e - 9$, we have $\mathrm{gd}(2^{e+1}\xi_n) \le \max\{d_{e,i} + d_{e,n-i} - 1 : 0 \le i \le n\}$. We must show that each $d_{e,i} + d_{e,n-i} - 1$ is $\le$ either $g(e+1,[\frac{n}{8}])$ or $2^e - \rho(e) - 1$. For those i such that $d_{e,i} = 2^e - \rho(e)$, we have $d_{e,n-i} = 0$, and so the desired result is true in these cases. For other i, the numbers $d_{e,i} - d_{e,n-i} - 1$ are among those which yielded $\mathrm{gd}(2^{e+1}\xi_n) \le g(e+1,[\frac{n}{8}])$ in 2.7.(6), yielding the claim in these cases. Finally, using (1), 2.5, and 2.6,

$$\mathrm{gd}(2^{e+1}\xi_{2^e-1}) \le \max\{2^e - \rho(e) - 1, g(e,\ell) + g(e, 2^{e-3} - 1 - \ell) - 1 : [\tfrac{\rho(e)}{8}] \le \ell \le 2^{e-3} - 1 - [\tfrac{\rho(e)}{8}]\}.$$

We have

$$g(e,\ell) + g(e, 2^{e-3} - 1 - \ell) = 2^e - 2e + (\delta + s)(\ell) + (\delta + s)(2^{e-3} - 1 - \ell),$$

which for $2 \le \ell \le 2^{e-3} - 3$ and $e \ge 7$ has maximum value of $2^e - e - 6$ when $\ell = 2^{e-4}$.

We will first prove (3) using f_0 instead of f, and then explain why it still holds when f is used. We wish to prove

$$d_{e,i} + \widetilde{\text{gd}}(2^{e+1}\xi_{n-i}) \le d'_{e,n}, \tag{6.4}$$

where $\widetilde{\text{gd}}(2^{e+1}\xi_{n-i})$ refers to the bound given in (2).

If $n < 2^e - \rho(e)$, there are two cases depending on whether or not $g(e+1, [\frac{n}{8}]) = g(e, [\frac{n}{8}])$. This equality occurs only for the bottom few nonzero elements in columns in (the extension of) Table 2.8 for which the column number is not divisible by 8. If $g(e+1, [\frac{n}{8}]) < g(e, [\frac{n}{8}])$, then

$$\begin{aligned} d_{e,i} + \widetilde{\text{gd}}(2^{e+1}\xi_{n-i}) &= g(e, [\tfrac{i}{8}]) + g(e+1, [\tfrac{n-i}{8}]) \\ &\le g(e, [\tfrac{i}{8}]) + g(e, [\tfrac{n-i}{8}]) \\ &\le g(e+1, [\tfrac{n}{8}]) + 1 \\ &= g(e, [\tfrac{n}{8}]) = d'_{e,n}. \end{aligned} \tag{6.5}$$

Here we used 2.7.(3) at the middle step. If, on the other hand, $g(e+1, [\frac{n}{8}]) = g(e, [\frac{n}{8}])$, then $g(e, j) = g(e+1, j) = 0$ for all $j < [\frac{n}{8}]$. (See Table 2.8.) Thus in this case $d_{e,i} + \widetilde{\text{gd}}(2^{e+1}\xi_{n-i}) \le g(e, [\frac{n}{8}])$, since at least one term is 0.

Now assume $2^e - \rho(e) \le n \le 2^e - 9$. (a) If $2^e - \rho(e) \ge g(e, [\frac{n}{8}])$, then $d_{e,i} + \widetilde{\text{gd}}(2^{e+1}\xi_{n-i}) \le g(e, [\frac{n}{8}])$ as in the previous paragraph, and this is $\le 2^e - \rho(e)$, as claimed. (b) The case in which $2^e - \rho(e) < g(e, [\frac{n}{8}])$ requires a little more argument. If $g(e+1, [\frac{n-i}{8}]) < g(e, [\frac{n-i}{8}])$, then the desired inequality follows similarly to (6.5). The first $\le$ there becomes $<$, and so we deduce $d_{e,i} + \widetilde{\text{gd}}(2^{e+1}\xi_{n-i}) \le g(e, [\frac{n}{8}]) - 1 = d'_{e,n}$. If, on the other hand, $g(e+1, [\frac{n-i}{8}]) = g(e, [\frac{n-i}{8}])$, then $g(e, k)$ with $k = [\frac{n-i}{8}]$ must be one of the equal bottom nonzero entries in a column $k \ge 2$, with $e = 4k - 1 + r$ with $\nu(k) \le r \le 2$. Then (6.4) becomes

$$4k + 4 \le g(e, 2^{e-3} - t) - 1 - g(e, 2^{e-3} - t - k) \tag{6.6}$$

with $[\frac{n}{8}] = 2^{e-3} - t$. The hypothesis $2^e - \rho(e) < g(e, 2^{e-3} - t)$ implies $8t - \nu(t) < e + 5 \le 4k + 6$. This implies $\nu(t + k) \le \lg(2k)$ and so the RHS of (6.6) is $\ge 8k - 1 - \lg(2k)$. Since $4k + 4 \le 8k - 1 - \lg(2k)$ for $k \ge 2$, (6.6) is valid, hence so is (6.4) in this case. Note that the inequalities in this paragraph are quite crude, but are all that we need here.

Finally, suppose $2^e - 8 \le n \le 2^e - 1$. We have

$$d_{e,0} + \widetilde{\mathrm{gd}}(2^{e+1}\xi_{2^e-1}) = 2^e - e - 7.$$

Cases in which $d_{e,i} = 2^e - \rho(e)$ have $\widetilde{\mathrm{gd}}(2^{e+1}\xi_{n-i}) = 0$, and, since $2^e - \rho(e) < 2^e - e - 7$, (6.4) is valid in these cases. In other cases, the LHS of (6.4) equals

$$g(e,\ell) + g(e+1, 2^{e-3} - 1 - \ell) = 2^e - 2e - 1 + \delta(\ell) + \delta(2^{e-3} - 1 - \ell).$$

The largest value of this occurs when $\ell = 2^{e-4}$ and is $2^e - e - 7$.

What remains is to show that incorporating positive values of s cannot affect validity of the above argument. We saw in the paragraph containing (5.12) that 2.7.(3), which is the primary tool throughout this proof, is valid with s incorporated. The above argument also required that $f(2^{e-3}-1) = f_0(2^{e-3}-1)$; i.e., that $s(2^{e-3}-1) = 0$. This is clear from Lemma 6.7, which implies that if $s(n) > 0$ then n has at least one huge gap (i.e. string of 0's) in its binary expansion, where "huge" is one with a number of 0's nearly eight times as large the value of the number which follows it. ■

We close with a complete account of how $s(n)$ can be nonzero in 5.4.

Lemma 6.7. *Suppose $s(n) = 1$.*

(1) *If this is due to $f_0(n) = f(n-1) + 1$ in 5.4, then either $n = A_0 + A_1$ with $\nu(A_0) = 8A_1 + \delta(n)$, or for some $t > 1$, $n = A_0 + A_1 + \cdots + A_t$ with $\nu(A_0) = 8A_1 + \nu(A_1) - 1$, $\nu(A_i) = 8A_{i+1} + \nu(A_{i+1}) - 2$ for $1 \le i < t-1$ and $\nu(A_{t-1}) = 8A_t + \delta(n) - 1$.*

(2) *If this is due to $f_0(n) = f(n-1) - 1$ in 5.4, then $n = n_* + B$ with n_* as in (1) and $\nu(A_t) \ge 8B + 3$.*

Example 6.8. *We illustrate this with the next-to-last example from 5.5. We have $s(2^{2^{20}+16} + 2^{17} + \epsilon) = 1$ as follows:*

- *If $\epsilon = 0$, it is type 6.7.(1) with $A_0 = 2^{2^{20}+2^{16}}$ and $A_1 = 2^{17}$.*
- *If $\epsilon = 1$, it is type 6.7.(2) with $n_* = A_0 + A_1$ as in the case $\epsilon = 0$ just considered, and $B = 1$.*
- *If $\epsilon = 2$, it is type 6.7.(1) with $A_0 = 2^{2^{20}+16}$, $A_1 = 2^{17}$, and $A_2 = 2$.*

Proof. (1) The inductive definition of s and f in 5.4, without regard for the specific definition of f_0, just the fact that f_0 is an increasing function, implies[5] that if $s(n) = 1$ due to $f_0(n) = f(n-1)+1$, then there is a positive integer t and integers $n_0 < \cdots < n_t = n$ such that

$$f_0'(n_i) - f_0'(n_{i-1}) = \begin{cases} 1 & i = 1 \\ 2 & 2 \le i \le t. \end{cases}$$

(It must also be true that if $n_{i-1} < m < n_i$, then $f_0'(m) < f_0'(n_i)$.)

Consider first the case $t = 1$. The difference $f_0'(m) - f_0'(m-1)$ is at least 5 $(= 8 + (-1) - 2)$ unless $\nu(m-1) \ge 4$. Thus the only way that $f_0'(n_1) - f_0'(n_0)$ might equal 1 is if $n_0 = u \cdot 2^e$ with $e \ge 4$ and $n = n_1 = u \cdot 2^e + A_1$ with $e = 8A_1 + \delta(n)$, using 5.4 and $\delta(n_0) = e - 1$. Note that A_0 in the lemma equals $u \cdot 2^e$. The claim of the lemma when $t = 1$ is thus established.

Now let $t = 2$. We must have $n_0 = A_0$ and $n_1 = A_0 + A_1$ as in the previous paragraph. If $n = n_2 = A_0 + A_1 + A_2$ with $f_0'(n_2) = f_0'(n_1) + 2$, then

$$8A_2 + \delta(n) = \delta(n_1) + 2.$$

This implies that $\delta(n_1) > 3$ and hence $\delta(n_1) = \nu(A_1) - 1$ by 5.3. This yields the claim $\nu(A_1) = 8A_2 + \delta(n) - 1$ of the lemma when $t = 2$. Note that the condition $\nu(A_0) = 8A_1 + \delta(A_0 + A_1)$ will now be given in the more explicit form with $\delta(A_0 + A_1)$ replaced by $\nu(A_1) - 1$, since we now have the additional information that $\nu(A_1) > 3$.

We will conclude with the case $t = 3$, after which the pattern for larger values of t will have become clear. We must have $n_0 = A_0$, $n_1 = A_0 + A_1$, and $n_2 = A_0 + A_1 + A_2$ as in the previous paragraph. If $n = n_2 + A_3$ satisfies $f_0'(n_3) = f_0'(n_2) + 2$, then

$$8A_3 + \delta(n) = \delta(n_2) + 2.$$

This implies that $\delta(n_2) > 3$ and hence $\delta(n_2) = \nu(A_2) - 1$, and yields the claim $\nu(A_2) = 8A_3 + \delta(n) - 1$ of the lemma when $t = 3$. The condition $\nu(A_1) = 8A_2 + \delta(A_0 + A_1 + A_2) - 1$ has $\delta(A_0 + A_1 + A_2)$ replaced by $\nu(A_2) - 1$.

(2) Cases of $s(n) = 1$ due to $f_0(n) = f(n-1) - 1$ must be caused by an n_* as in (1) with $n = n_* + B$ and $f_0'(n_* + i) \le f_0'(n_*)$ for $1 \le i \le B$. This can

[5] One can formulate and prove a closely-related result about how to form from a strictly increasing sequence $\langle n_i \rangle$ of integers an increasing sequence $\langle m_i := n_i + s_i \rangle$ with $s_i = 0$ or 1 such that $m_{i+1} - m_i$ never equals 1.

only happen if $8B + \delta(n_* + B) \le \delta(n_*)$. As this implies that $\delta(n_*) > 3$, we have $\delta(n_*) = \nu(n_*) - 1 = \nu(A_t) - 1$. Because n_* contains large gaps, we must also have $\delta(n_* + B) \ge 2$, and hence $\nu(A_t) \ge 8B + 3$. ■

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