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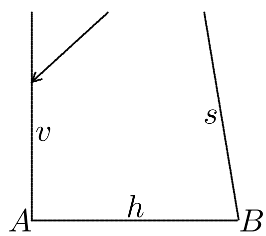
1. If $\frac{1}{3} + \frac{1}{4} = \frac{1}{x}$, then $x =$
2. A gold bar is a rectangular solid measuring $2 \times 3 \times 4$. It is melted down, and three equal cubes are constructed from this gold. What is the length of a side of each cube?
3. What is the area of a triangle whose sides have length 10, 13, and 13?
4. The perimeter of a rectangle is 28. A second rectangle is three times as long as the first, and twice as wide. The perimeter of the second rectangle is 72. What is the area of the first rectangle?
5. A lamppost is 20 feet high. How many feet away from the base of the post should a person who is 5 feet tall stand in order to cast an 8-foot shadow?
6. Workers in an office have a coffee machine and make $1\frac{1}{2}$ cents profit on each cup sold. The profits were split at the end of the year, and everyone got \$2.50, with a total of \$25 remaining after that. It was suggested that instead everyone should get \$3, but that would have left three people with no money. How many cups of coffee were sold?
7. In a circle whose diameter is 30, a chord is drawn perpendicular to a radius. The distance from the point where the chord intersects the radius to the outer end of the radius is 3. What is the length of the chord?
8. What is the smallest number of coins (possible denominations 1, 5, 10, 25, and 50 cents) which will enable one to pay the exact price for any item costing from 1 cent up to and including one dollar (100 cents)?
9. What is the solution set of the inequality $x^3 + x^2 - 2x \geq 0$?
10. In the diagram below, line v is perpendicular to line h, and the angle at B is 75 degrees. A light ray has angle of reflection equal to angle of incidence. Our convention is to consider the angle which is ≤ 90 degrees. A ray has initial angle of incidence with v of 50 degrees. It follows a path in which it next hits wall h, then s, and then v again. What will be the number of degrees of its angle of incidence with wall v this (second) time?



11. What is the smallest positive integer p for which there is a positive integer n satisfying $2^{n+p} \equiv 2^n \bmod 100$? (This p is the *period* of the sequence $\langle 2^n \bmod 100 \rangle$, which just considers the last two digits of the numbers.)
12. A polynomial has remainder 2 when divided by $x - 1$, and has remainder 1 when divided by $x - 2$. What remainder is obtained when this polynomial is divided by $(x - 1)(x - 2)$?

13. Write the decimal approximation of $\sqrt{9{,}000{,}001} - \sqrt{9{,}000{,}000}$, rounded to three significant figures.

14. What is the largest integer n such that $20!$ is divisible by 80^n? (Note $20! = 1{\cdot}2{\cdot}3 \cdots 20$.)

15. For what positive integer b does the number whose base-b expansion is 265 equal the number whose base-9 expansion is $1b1$?

16. The cross-sectional area of a tree is a linear function of time. The diameter is 2 feet in 1930 and 4 feet in 1990. What is its diameter in feet in 2010?

17. A quadrilateral is inscribed inside a circle. If an angle is now inscribed in each of the four arcs outside the quadrilateral but inside the circle, what is the sum of the number of degrees in these four angles?

18. In a certain town, there are 5000 bicycles, each of which is assigned a license number from 1 to 5000. No two bicycles receive the same number. What is the probability that the number on a randomly selected bicycle will not have any 8's among its digits? (You may write your answer as a fraction or a decimal number.)

19. Let $f(x) = \frac{x}{x+3}$ for $x \neq -3$. List all values of x for which $f(f(x)) = x$.

20. What is the coefficient of x^6 in the expansion of $(2x^2 - \frac{1}{x})^6$?

21. For how many integers x between 1 and 91 inclusive is it true that $x^2 - 3x + 2$ is a multiple of 91?

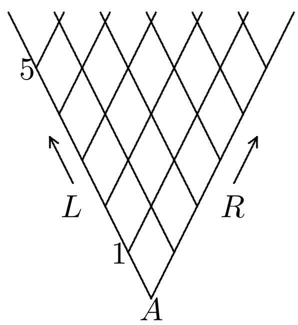
22. Two roots of the polynomial $3x^3 + \alpha x^2 - 5x - 10$ are r and $-r$ for some real number r. What is the value of α?

23. Let $A = (\sin(.01))^2$, $B = (\sin(.01))^3$, $C = \sin(\sin(.01))$, and $D = \sin(\sin(\sin(.01)))$, where the .01 refers to radian measure. Write the letters A, B, C, and D in order of size, starting with the smallest and ending with the largest of the four.

24. A point P is chosen inside an equilateral triangle of side length 1, and perpendiculars drawn to the three sides. Let $S(P)$ denote the sum of the lengths of the three perpendiculars. What is the largest possible value for $S(P)$ out of all points P inside the triangle?

25. What is the smallest positive integer N such that both $\sqrt{N/3}$ and $\sqrt[3]{N/2}$ are integers?

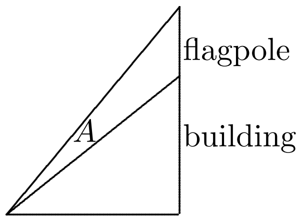
26. The bottom part of a network of roads is shown below. It extends up to level 99 in a similar fashion. 2^{100} people leave point A (level 0). At every intersection, half go in direction L and half in direction R. How many people will end up at the next-to-left position at level 99?



27. In how many ways can the faces of a cube be colored using six colors, if each face is to be a different color, and two colorings are considered the same when one can be obtained from the other by rotating the cube?

28. An ordinary sheet of graph paper has lines wherever x or y is an integer. How many paths are there from $(0, 0)$ to $(3, 4)$ which always move along lines in either the positive x or y direction?

29. A flagpole is on top of a building which is 40 feet high. From a point 50 feet away from the building, the flagpole subtends an angle A whose tangent is $1/5$. How many feet tall is the flagpole?



30. The perimeter of a right triangle is 30, and the length of the altitude perpendicular to the hypotenuse is 5. What is the length of the hypotenuse?

31. A convex polygon with n sides has all angles equal to 150 degrees, with the possible exception of one angle. List all the possible values of n.

32. Alan and Bill walk at 4 miles per hour, and Chad drives 40 miles per hour. They set out from the same point in the same direction with Alan walking, and Bill riding with Chad. After an hour, Bill gets out of the car and starts walking in the same direction they were going, while Chad turns the car around and drives back to pick up Alan. When the car gets back to Alan, Alan gets into the car, the car turns around and drives until it meets Bill. How many miles has Bill traveled when the car meets him? You may assume that no time is required to turn around or to change passengers.

33. List all prime numbers which are of the form $x^3 - 11x^2 - 107x + 1177$ for some integer x.

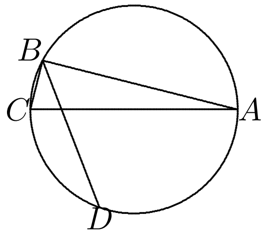
34. A *Fibo* sequence $a_1, a_2, \dots$ is one in which a_1 and a_2 are positive integers, and $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$. If $a_1, a_2, \dots, 81, \dots$ is the Fibo sequence which contains 81 and has the largest possible number of terms preceding the 81, write the ordered pair (a_1, a_2).

35. What fourth degree polynomial p has $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$ equal to 7, 1, 3, 1, and 7, respectively?

36. Four spheres of radius 1 are placed so that each is tangent to the other three. What is the radius of the smallest sphere that contains all four spheres?

37. Evaluate $\sum_{n=1}^{999} [.729n]$, where $[q]$ denotes the greatest integer $\leq q$.

38. Points A, B, C, and D lie on a circle. AC is a diameter, and $\angle CBD = \angle DBA$. If $BC = 2$ and $AB = 4$, what is the length of BD? The diagram below is not quite to scale.



39. How many ways are there to give 18 indistinguishable cookies to Alice, Bob, Carol, Don, and Elizabeth in such a way that everyone gets at least one, and no one gets more than 6?

40. List all real values of p for which the two solutions, α and β, of the equation

$$x^2 - 2px + p^2 - 2p - 1 = 0$$

have the property that $((\alpha - \beta)^2 - 2)/(2((\alpha + \beta)^2 + 2))$ is an integer.

SOLUTIONS, annotated with number of people out of the 46 people who scored at least 22 answering it correctly.

1. 12/7. [46] $\frac{4}{12} + \frac{3}{12} = \frac{7}{12}$.
2. 2. [46] The volume of the solid was 24, which will be divided into three cubes, each of volume 8. They will have side length $\sqrt[3]{8}$.
3. 60. [45] The triangle is isosceles, and has a base of 10 and a height of $\sqrt{13^2 - 5^2} = 12$.
4. 48. [45] $L + W = 14$ and $3L + 2W = 36$. We solve these to find $L = 8$ and $W = 6$.
5. 24. [44] If x is the desired distance, then, by similar triangles, $5/20 = 8/(8 + x)$.
6. 13,000. [36] If N is the number of people, then $\frac{5}{2}N + 25 = 3(N - 3)$, and so $N = 68$. Thus the profit was 195, and the number of cups was $195/.015 = 13{,}000$.
7. 18. [45] Let O be the center of the circle, OP the radius which is perpendicular to the chord, R the point where the chord intersects the radius, and Q one end of the chord. Then $OR = 15 - 3$, and ORQ is a right triangle with hypotenuse $OQ = 15$ and one leg $OR = 12$. Its other side, RQ equals 9, and is half the length of the chord.
8. 9. [32] The coins would be 1, 1, 1, 1, 5, 10, 10, 25, 50. The first six coins are required to get you up to 19 in the most efficient way. You must add another 10 to get to 25, and then the 25 and 50 cover all other cases most efficiently.
9. $[-2, 0] \cup [1, \infty)$ or $(-2 \leq x \leq 0$ or $x \geq 1)$. [44] The inequality is $x(x + 2)(x - 1) \geq 0$. All three or exactly one of the factors must be nonnegative.
10. 80. [41] If it hits h at C, the angle there is 40 degrees. Then it hits s at point D with angle $180 - 75 - 40 = 65$. If it next hits v at point E, then $ABDE$ is a quadrilateral with three of its angles 90, 75, and $(180 - 65)$. Since a quadrilateral has 360 degrees, this makes the desired angle $360 - 280 = 80$.
11. 20. [40] Start with 1 and keep doubling, but remove all but the last two digits. Keep doing it until a repeat occurs. 2, 4, 8, 16, 32, 64, 28, 56, 12, 24, 48, 96, 92, 84, 68, 36, 72, 44, 88, 76, 52, 4. Now it keeps repeating. There were 20 steps from one 4 to the next. We have $2^{n+20} \equiv 2^n \bmod 100$ for $n \geq 2$.
12. $-x + 3$. [32] Write the polynomial as $q(x)(x - 1)(x - 2) + ax + b$. Then $a + b = 2$ and $2a + b = 1$, leading to $a = -1$ and $b = 3$.
13. .000167. [20] Multiply by S/S, where $S = \sqrt{9{,}000{,}001} + \sqrt{9{,}000{,}000}$, which is very close to 6000. We obtain $1/S$.
14. 4. [43] $80^n = 2^{4n}5^n$, and 20! has exponent of 2 equal to $10 + 5 + 2 + 1 = 18$ and exponent of 5 equal to 4.
15. 7. [45] We have $2b^2 + 6b + 5 = 81 + 9b + 1$, hence $2b^2 - 3b - 77 = 0$, which can be solved either by factoring or by the quadratic formula.

16. $2\sqrt{5}$. [40] The area grew from π to 4π in 60 years. In the next 20 years it will increase by an additional $3\pi/3$. Thus the area will be 5π, so the radius will be $\sqrt{5}$, and the diameter $2\sqrt{5}$.

17. 540. [33] If APB is one of these angles, then the number of degrees in angle P equals $1/2$ times the number of degrees in the arc AB which does not include P. All portions of the circle will be included in three of the four of these arcs. Therefore the answer is $\frac{1}{2} \cdot 3 \cdot 360$.

18. .729. [32] The first digit is definitely not an 8, while for each of the other three digits, the probability that it is not an 8 is 0.9. The answer is $.9^3$.

19. 0, -2. [40] We must have

$$\frac{x/(x+3)}{\frac{x}{x+3}+3} = x.$$

This simplifies to $x = x(4x+9)$, which is true for $x = 0$ or -2.

20. 240. [41] The x^6-term is $\binom{6}{4}(2x^2)^4(-\frac{1}{x})^2$, so the coefficient is $2^4 15$.

21. 4. [21] Since $x^2 - 3x + 2 = (x-1)(x-2)$, both $x = 1$ and $x = 2$ make the expression equal to 0, which is a multiple of 91. In addition to these, we look for multiples of 13 for which adding or subtracting 1 yields a multiple of 7. This works for 13 and 78.

22. 6. [40] We must have $3x^3 + \alpha x^2 - 5x - 10 = (x^2 - r^2)(3x + \frac{10}{r^2})$. The second factor is chosen to make the constant and cubic terms work out. Thus $\alpha = \frac{10}{r^2}$ and $-5 = -3r^2$. Hence $\alpha = 10/(5/3)$.

23. BADC. [25] $(\sin(.01))^3 \approx .000001$, $(\sin(.01))^2 \approx .0001$. The other two are approximately .01, but $\sin(x) < x$ for $x > 0$.

24. $\frac{1}{2}\sqrt{3}$. [41] $S(P)$ has the same value for all points P. The lines from P to the vertices of the triangle divide the triangle into three subtriangles. The area of each subtriangle is $\frac{1}{2}h$, where h is its altitude. The total area of the triangle, $\frac{1}{4}\sqrt{3}$, equals the sum of the areas of the three subtriangles, which is $\frac{1}{2}(h_1 + h_2 + h_3)$. The sum which we desire is $h_1 + h_2 + h_3$.

25. 432. [44] Since extraneous factors are to be avoided, N should be of the form $2^a 3^b$. The first requirement forces a to be even and b to be odd. The second requirement forces $a \equiv 1 \bmod 3$ and $b \equiv 0 \bmod 3$. Thus $a = 4$ and $b = 3$ is the smallest possibility, and $N = 16 \cdot 27 = 432$.

26. 198. [33] This is like Pascal's triangle. At level n, the fraction of the people in horizontal position i is $\binom{n}{i}/2^i$, and so the number of people in position 1 at level 99 is $\binom{99}{1}2^{100}/2^{99} = 198$.

27. 30. [22] Call the colors 1 through 6. There are 5 choices for the color opposite color 1. Then there are 3 choices for the color opposite the smallest color not yet used. The two ways of placing the remaining two colors yield inequivalent colorings. Thus the answer is $5 \cdot 3 \cdot 2$.

28. 35. [39] It is the binomial coefficient $\binom{7}{3}$ since there must be 7 steps of which exactly 3 move in the x direction.

29. $19\frac{11}{21}$ or $\frac{410}{21}$. [32] Let B be the angle to the top of the building.

$$\frac{x+40}{50} = \tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)} = \frac{\frac{1}{5}+\frac{4}{5}}{1-\frac{1}{5}\frac{4}{5}} = \frac{25}{21}.$$

Thus $x = \frac{1250}{21} - 40$.

30. $12\frac{6}{7}$ or $\frac{90}{7}$. [21] Let c denote the hypotenuse, and a and b the legs. Then $a+b+c = 30$, $5c = ab$, and $a^2+b^2 = c^2$. Eliminating c yields $30-a-b = \frac{ab}{5}$ and $(\frac{ab}{5})^2 = a^2+b^2$. Equating the two expressions for $(\frac{ab}{5})^2$ leads to $60a+60b = 30^2+2ab$, while the first says $60a+60b = 30\cdot 60 - 12ab$. Equating these yields $ab = 900/14$ and hence $c = \frac{90}{7}$.

31. 8,9,10,11,12. [32] The sum of the interior angles of a convex n-gon is $(n-2)180$. This must equal $(n-1)150+a$ with $0 < a < 180$. We obtain $a = 30(n-7)$, Thus $7 < n < 13$. It is easy to check that each of these convex n-gons can be formed.

32. $47\frac{3}{11}$ or $\frac{520}{11}$. [18] Let t denote the number of hours during which Chad was driving back to pick up Alan. Then $4(1+t) = 40-40t$, hence $t = \frac{9}{11}$. Although you can compute it by solving equations, you can also reason it out that Bill will have to drive forward the second time for exactly one hour in order to catch up with Bill. The reason for this is that if we forget about the time period when they were both walking, we see that when they eventually meet, they both will have been in the car the same amount of time. So, during the two hours when one was walking and one riding each will have traveled 44 miles, and during the other 9/11 hour, each will have traveled 36/11 miles.

33. 7, 37. [21] $(x-11)(x^2-107)$ is composite or 0 except possibly when $x-11 = \pm 1$ or $x^2-107 = \pm 1$. Since 106 and 108 are not perfect squares, the latter does not happen for integer x. If $x = 10$ or 12, then the product equals 7 or 37.

34. $(3,2)$. [25] Suppose the term immediately before 81 is x. Then, working backwards from there, the next terms are $81-x$, $2x-81$, $162-3x$, $5x-243$, $405-8x$, $13x-648$, $1053-21x$, and $34x-1701$. To make these be all positive, we must have $x \le 80$, $x \ge 41$, $x \le 53$, $x \ge 49$, $x \le 50$, $x \ge 50$, $x \le 50$, and $x \ge 51$. The last inequality is inconsistent with those which precede it. Hence $x = 50$, and the sequence is 3, 2, 5, 7, 12, 19, 31, 50, 81.

35. $x^4-8x^3+21x^2-20x+7$. [22] One way is to write $p(x) = ax^4+bx^3+cx^2+dx+7$ and solve four linear equations for a, b, c, and d. A better way is to note that if $q(x) = p(x+2)$, then $q(\pm 1) = 1$ and $q(\pm 2) = 7$, and so $q(x) = ex^4+fx^2+3$. We get $e+f = -2$ and $16e+4f = 4$, from which we obtain $e = 1$ and $f = -3$. Thus $p(x) = q(x-2) = (x-2)^4 - 3(x-2)^2 + 3$, which is easily expanded to the claimed polynomial.

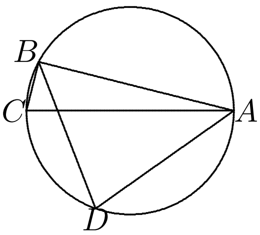
36. $1+\frac{1}{2}\sqrt{6}$. [10] The centers of the four spheres form a regular tetrahedron of side length 2. The center of the the bounding sphere must be at the point in the tetrahedron which

is the intersection of its altitudes. Each altitude will have one end at the centroid of a face, so will have length $\sqrt{2^2-(\frac{2}{3}\sqrt{3})^2}=\sqrt{8/3}$. Just as the intersection of the altitudes of an equilateral triangle is 1/3 of the way up the altitudes, the intersection of the altitudes of a regular tetrahedron is 1/4 of the way up the altitudes. Thus the distance from the center of the bounding sphere to one of the other centers is $\frac{3}{4}\sqrt{8/3}=\frac{1}{2}\sqrt{6}$. We add 1 to this to get the radius of the bounding sphere.

37. 363636. [6] This is the case $a=1000$ and $b=729$ of the following, which has $(a-1)(b-1)/2$ as the answer. Evaluate $\sum_{n=1}^{a-1}[\frac{b}{a}n]$ if a and b have no common factors. The answer is best obtained by thinking of the grid along integer lines in the rectangle $[0,a]\times[0,b]$. Since a and b have no common factors, the diagonal from $(0,0)$ to (a,b) will not pass through any lattice points, except at its endpoints. Thus the desired answer is the number of lattice points having $y>0$ lying below the line which satisfy $1\le n\le a-1$. This diagonal will exactly bisect the rectangle $[1,a-1]\times[0,b]$. The total number of lattice points in this rectangle, excluding the top and bottom edges, is $(a-1)(b-1)$, and half of them lie below the line.

38. $3\sqrt{2}$. [11] Since AC is a diameter, B is a right angle, and so angle CBD and DBA are 45 degrees. Angle CAD is also 45 degrees ($\pi/4$ radians), since it subtends the same arc as does angle CBD. We have $AC=\sqrt{2^2+4^2}=2\sqrt{5}$, and, if $\theta=\angle BAC$, then $\sin\theta=1/\sqrt{5}$. By the Extended Law of Sines, $BD/\sin(\angle BAD)$ equals the diameter of the circle. Thus

$$BD=2\sqrt{5}\sin(\theta+\tfrac{\pi}{4})=2\sqrt{5}\tfrac{\sqrt{2}}{2}(\sin(\theta)+\cos(\theta))=\sqrt{10}\tfrac{3}{\sqrt{5}}=3\sqrt{2}.$$



39. 780. [3] After giving one to each, there are 13 left to distribute. You can enumerate the number of distributions directly, by considering all the ways to have five numbers add up to 13, and multiply each possibility by the appropriate permutation number. But generating functions provide a better method. Let $p(x)=1+x+\cdots+x^5$. Then the monomials in $p(A)p(B)p(C)p(D)p(E)$ whose exponents sum to 13 correspond to the desired ways. The number of these is the coefficient of x^{13} in

$$p(x)^5=\left(\frac{1-x^6}{1-x}\right)^5=(1-5x^6+10x^{12}-\cdots)(1-x)^{-5}.$$

Since the coefficient of x^n in $(1-x)^{-5}$ is $\binom{n+4}{4}$, the answer is $10\binom{5}{4}-5\binom{11}{4}+\binom{17}{4}=50-5\cdot 330+2380=780$.

40. $-\frac{1}{4}$, $\frac{1}{2}$. [3] We have $\alpha+\beta=2p$ and $\alpha\beta=p^2-2p-1$. Thus

$$(\alpha-\beta)^2=(2p)^2-4(p^2-2p-1)=4(2p+1).$$

We want real values of p for which $\frac{4p+1}{4p^2+2}=n$, with n an integer. If $n=0$, we get $p=-\frac{1}{4}$. If $n\neq 0$, this yields a quadratic equation for p, with solution

$$p=\frac{1\pm\sqrt{1-n(2n-1)}}{2n}.$$

This gives a real solution for p only if $-\frac{1}{2}\leq n\leq 1$. The only nonzero integer n in this range is 1, yielding $p=\frac{1}{2}$. (Although not necessary for solving the problem, one can find that for $p=-\frac{1}{4}$, $\{\alpha,\beta\}=\{(-1\pm 2\sqrt{2})/4\}$ and if $p=\frac{1}{2}$, then $\{\alpha,\beta\}=\{(1\pm 2\sqrt{2})/2\}$.)