TOPOLOGICAL COMPLEXITY OF PLANAR POLYGON SPACES WITH SMALL GENETIC CODE

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Abstract. We determine, within 1, the topological complexity of many planar polygon spaces mod isometry. In all cases considered except for those homeomorphic to real projective space or n-tori, the upper and lower bounds given by dimension and cohomology considerations differ by 1. The spaces which we consider are those whose genetic codes, in the sense of Hausmann and Rodriguez, have a single gene, and its size is ≤ 4.

1. Statement of results

The topological complexity, $\mathrm{TC}(X)$, of a topological space X is, roughly, the number of rules required to specify how to move between any two points of X. A "rule" must be such that the choice of path varies continuously with the choice of endpoints. (See [3, §4].) We study $\mathrm{TC}(X)$ where $X = \overline{M}(\bar\ell)$ is the space of equivalence classes of oriented n-gons in the plane with consecutive sides of length $\ell_1, \ldots, \ell_n$, identified under translation, rotation, and reflection. (See, e.g., [4, §6].) Here $\bar\ell = (\ell_1, \ldots, \ell_n)$ is an n-tuple of positive real numbers. Thus

$$\overline{M}(\bar\ell) = \{(z_1, \ldots, z_n) \in (S^1)^n : \ell_1 z_1 + \cdots + \ell_n z_n = 0\}/O(2).$$

We can think of the sides of the polygon as linked arms of a robot, and then $\mathrm{TC}(X)$ is the number of rules required to program the robot to move from any configuration to any other.

Since permuting the ℓ_i's does not affect the space up to homeomorphism, we may assume $\ell_1 \le \cdots \le \ell_n$. We assume that $\ell_n < \ell_1 + \cdots \ell_{n-1}$, so that the space $\overline{M}(\bar\ell)$ has more than one point. We also assume that $\bar\ell$ is *generic*, which means that there is

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no subset $M \subset [n] = \{1, \ldots, n\}$ with $\sum_{i \in M} \ell_i = \frac{1}{2} \sum_{i=1}^{n} \ell_i$. When $\bar{\ell}$ is generic, $\overline{M}(\bar{\ell})$ is an $(n-3)$-manifold ([4, p.314]), and hence, by [3, Cor 4.15], satisfies

$$\mathrm{TC}(\overline{M}(\bar{\ell})) \leq 2n - 5. \tag{1.1}$$

It is well-understood that the homeomorphism type of $\overline{M}(\bar{\ell})$ is determined by which subsets S of $[n]$ are *short*, which means that $\sum_{i \in S} \ell_i < \frac{1}{2} \sum_{i=1}^{n} \ell_i$. For generic $\bar{\ell}$, a subset which is not short is called *long*. Define a partial order on the power set of $[n]$ by $S \leq T$ if $S = \{s_1, \ldots, s_\ell\}$ and $T \supset \{t_1, \ldots, t_\ell\}$ with $s_i \leq t_i$ for all i. As introduced in [5], the *genetic code* of $\bar{\ell}$ is the set of maximal elements (called *genes*) in the set of short subsets of $[n]$ which contain n. The homeomorphism type of $\overline{M}(\bar{\ell})$ is determined by the genetic code of $\bar{\ell}$. Note that if $\bar{\ell} = (\ell_1, \ldots, \ell_n)$, then all genes have largest element n.

Our main theorem is

Theorem 1.2. *If the genetic code of $\bar{\ell}$ consists of a single gene of size* 2, 3, *or* 4, *with largest element $n \geq 5$, and is not $\langle\{5, 2, 1\}\rangle$ or $\langle\{6, 3, 2, 1\}\rangle$, then* $\mathrm{TC}(\overline{M}(\bar{\ell})) \geq 2n - 6$.

The two excluded cases are homeomorphic to tori $(S^1)^2$ and $(S^1)^3$. It is known ([3, (4.12)]) that $\mathrm{TC}((S^1)^n) = n + 1$; its genetic code is $\langle\{n, n-3, n-4, \ldots, 1\}\rangle$. The other known case in which it is not true that $\mathrm{TC}(\overline{M}(\bar{\ell})) \geq 2n - 6$ is the case in which the genetic code is $\langle\{n\}\rangle$. This space $\overline{M}(\bar{\ell})$ is homeomorphic to RP^{n-3}, for which the TC is often much less than $2n - 7$.([3, §4.8]) In [1], we showed that $\mathrm{TC}(\overline{M}(1^{n-1}, n - 2k)) \geq 2n - 6$ if $2 < 2k < n$. (We use exponents for repetition in length vectors.) The genetic code of this is $\langle\{n, n-1, \ldots, n-k+1\}\rangle$.

Thus in all cases considered so far, except for tori and real projective spaces, the upper bound for topological complexity of these planar polygon spaces given by dimension and the lower bound given by our cohomological argument differ by 1. This includes 12 out of the 21 6-gon spaces but only 26 out of the 135 7-gon spaces.([6]) Only the complicated nature of the cohomology rings $H^*(\overline{M}(\bar{\ell}); \mathbb{Z}_2)$ hinders investigation of spaces with more complicated genetic codes. It seems not unreasonable to think that perhaps all spaces $\overline{M}(\bar{\ell})$ of n-gons satisfy $\mathrm{TC}(\overline{M}(\bar{\ell})) \geq 2n - 6$ except for real projective spaces and tori.

We will prove in Theorems 2.3, 2.5, and 3.24 that, if $X = \overline{M}(\bar\ell)$ with single gene of size 2, 3, and 4, respectively, and with largest element n, then there are classes $v_1, \ldots, v_{2n-7}$ in $H^1(X)$ such that

$$\prod_{i=1}^{2n-7} (v_i \times 1 + 1 \times v_i) \neq 0 \in H^*(X \times X). \tag{1.3}$$

Here and throughout, all cohomology groups have coefficients in $\mathbb{Z}_2$. Theorem 1.2 then follows from [3, Cor 4.40].

2. Proof of Theorem 1.2 for genetic codes with a single gene of size 2 and 3

The first two of the three cases of (1.3) are handled in this section.

We review the determination of $H^*(\overline{M}(\bar\ell))$ obtained in [4], similar to the interpretation in [1, Thm 2.1].

Proposition 2.1. *If $\bar\ell$ has length n, then the ring $H^*(\overline{M}(\bar\ell))$ is generated by classes $R, V_1, \ldots, V_{n-1}$ in $H^1(\overline{M}(\bar\ell))$ subject to only the following relations:*

1. *All monomials of the same degree which are divisible by the same V_i's are equal. If S denotes this set of i's and d the degree, this element is denoted by T_S^d.*
2. *If $S \subset [n-1]$ has $S \cup \{n\}$ long, then $\prod_{i \in S} V_i = 0$.*
3. *If $L \subset [n-1]$ is long and $|L| \leq d+1$, then*

$$\sum_{\substack{S \subset L \\ S \cup \{n\} \text{ short}}} T_S^d = 0. \tag{2.2}$$

Note that the d is not an exponent, and that if $S = \emptyset$, then $T_S^d = R^d$ is included in the above sum. We will often denote $T_{\{i\}}^d$, $T_{\{i,j\}}^d$, and $T_{\{i,j,k\}}^d$ by T_i^d, $T_{i,j}^d$, and $T_{i,j,k}^d$, respectively, and will omit the superscript when it it clear from the context. If the LHS of (2.2) is denoted $\mathcal{R}_L^d$, then the fact that

$$V_\ell \mathcal{R}_L^d = \begin{cases} \mathcal{R}_{L \cup \{\ell\}}^{d+1} - \mathcal{R}_L^{d+1} & \ell \notin L \\ 0 & \ell \in L \end{cases}$$

and $R\mathcal{R}_L^d = \mathcal{R}_L^{d+1}$ implies that the relations (2.2) span the ideal which they generate.

The first case of (1.3), and hence Theorem 1.2, follows from the following result. We remark that one such $\overline{\ell}$ is $(1^a, 2^{n-a-1}, 2n-a-5)$.

Theorem 2.3. *If $\overline{\ell}$ has genetic code $\langle\{n, a\}\rangle$ with $1 \le a \le n-1$, and $X = \overline{M}(\overline{\ell})$, then*

$$(V_1 \times 1 + 1 \times V_1)^{n-3}(R \times 1 + 1 \times R)^{n-4} \ne 0 \in H^*(X \times X).$$

Proof. Since the genetic code has no sets of length 3, for all 2-subsets S, $S \cup \{n\}$ is long, and hence all products V_iV_j are 0. Since $\{n, i\}$ is long for $i > a$, we have $V_i = 0$ for $i > a$. The long subsets contained in $[n-1]$ are the complements of the short subsets containing n. These long subsets are exactly the $(n-2)$-sets $\{n-1, \ldots, \widehat{i}, \ldots, 1\}$, where the omitted element i satisfies $i \le a$. Thus the only relations of type (3) occur in degree $n-3$. Therefore, for $1 \le d \le n-4$, a basis for $H^d(X)$ is $\{R^d, T_1^d, \ldots, T_a^d\}$.

The subsets $S \subset \{n-1, \ldots, \widehat{i}, \ldots, 1\}$ for which $S \cup \{n\}$ is short are just $\emptyset$ and $\{j\}$ for $j \le a$ and $j \ne i$. Thus the relations of type (3) are

$$R^{n-3} + \sum_{\substack{j=1\\ j\ne i}}^{a} T_j^{n-3} = 0.$$

Subtracting pairs of relations reduces this set of relations to $T_1^{n-3} = \cdots = T_a^{n-3}$ and $R^{n-3} = (a-1)T_1^{n-3}$. Note that this implies $H^{n-3}(X) = \mathbb{Z}_2$, which must be true since X is an $(n-3)$-manifold.

Let $m = n-3$. We obtain, in bidegree $(m, m-1)$,

$$\begin{aligned} & (V_1 \times 1 + 1 \times V_1)^m (R \times 1 + 1 \times R)^{m-1} \\ = \ & \sum \tbinom{m}{i}\tbinom{m-1}{m-i} V_1^i R^{m-i} \times V_1^{m-i} R^{i-1} \\ = \ & (\tbinom{2m-1}{m} + 1) T_1^m \times T_1^{m-1} + T_1^m \times R^{m-1}. \end{aligned} \qquad (2.4)$$

Here we have used that $\sum \binom{m}{i}\binom{m-1}{m-i} = \binom{2m-1}{m}$ and noted that all terms in the sum are of the form $T_1^m \times T_1^{m-1}$ except the one with $i = m$. Since $\{R^{m-1}, T_1^{m-1}\}$ is linearly independent and $T_1^m \ne 0$, (2.4) is nonzero. ∎

Now let $X = \overline{M}(\overline{\ell})$ with genetic code $\{n, a+b, a\}$ with $n > a+b > a > 0$ and $n \ge 6$. Although irrelevant to our proof, we note this can be realized by $\overline{\ell} =$

$(2^a, 3^b, 6^{n-a-b-1}, 6n-3b-4a-17)$. For a class $v \in H^1(X)$, we will let $Dv = v\otimes 1+1\otimes v$. (Think "diagonal.") Note also that we are switching to tensor product notation, using the Künneth isomorphism. The second case of (1.3 and hence Theorem 1.2 follows immediately from the following result, the proof of which will occupy the rest of this section.

Theorem 2.5. *Let $X = \overline{M}(\bar{\ell})$ with genetic code $\{n, a+b, a\}$ with $n > a+b > a > 0$ and $n \geq 6$. Let $m = n-3$ and suppose $2^{e-1} < m \leq 2^e$. Then in $H^*(X \times X)$,*

a. *if a is even, then $(DV_1)^{2m-1-2^e}(DV_{a+b})(DR)^{2^e-1} \neq 0$;*
b. *if a is odd and $m \neq 2^{e-1}+1$, then $(DV_1)^{m-1}(DV_{a+b})^2(DR)^{m-2} \neq 0$;*
c. *if a is odd and $m = 2^{e-1}+1$, then $(DV_1)^m(DV_{a+b})^{m-1} \neq 0$.*

Similarly to the previous proof, Proposition 2.1 easily shows that, for $2 \leq d \leq n-5$, a basis for $H^d(X)$ is $\{R^d\} \cup \{T_i^d : 1 \leq i \leq a+b\} \cup \{T_{i,j}^d : i \leq a,\ i < j \leq a+b\}$, and all other T_S^d are 0. Indeed, the sets S such that $S \cup \{n\}$ is short are $\{j, i\}$ with $i < j \leq a+b$ and $i \leq a$, $\{i\}$ with $i \leq a+b$, and $\emptyset$. There are no additional relations until degree $n-4$ since the smallest $L \subset [n-1]$ for which $L \cup \{n\}$ is long has $|L| = n-3 = m$.

All the classes V_i with $i \leq a$ play the same role in the relations of type (3) in Proposition 2.1, as do all the classes V_i with $a < i \leq a+b$. The way that we will show a class z in $H^{2m-1}(X \times X)$ is nonzero is by constructing a uniform homomorphism $\psi : H^{m-1}(X) \to \mathbb{Z}_2$ such that $(\phi\otimes\psi)(z) \neq 0$, where $\phi : H^m(X) \to \mathbb{Z}_2$ is the Poincaré duality isomorphism. By *uniform homomorphism*, we mean one satisfying

- $\psi(T_i) = \psi(T_j)$ if $i, j \leq a$ or if $a < i, j \leq a+b$, and
- $\psi(T_{i,j}) = \psi(T_{i,k})$ if $j, k \leq a$ or if $a < j, k \leq a+b$.

We will let Y_1 refer to any T_i with $i \leq a$, and Y_2 to any T_i with $a < i \leq a+b$. Similarly, $Y_{1,1}$ denotes $T_{i,j}$ with $i < j \leq a$, while $Y_{1,2}$ is $T_{i,j}$ with $i \leq a < j \leq a+b$. Usually the superscript, indicating the grading, will be implicit. If $\theta : [a+b] \to \{1,2\}$ is defined by $\theta(i) = 1$ if $i \leq a$, and 2 otherwise, then $Y_{\theta(S)} = T_S$, where $\theta(S)$ is a multiset. If ψ is a uniform homomorphism, then $\psi(Y_W)$ is a well-defined element of $\mathbb{Z}_2$ for each of the four possible W. We also let $w_{\theta(i)} = V_i$. Thus V_1 and V_{a+b} in Theorem 2.5

are replaced by w_1 and w_2, respectively. Also, R will sometimes be denoted by w_0. Finally, an element of $[a+b]$ is of type 1 if it is $\le a$, and otherwise is of type 2.

The relations of type (3) in $H^{m-1}(X)$ have L obtained from $[n-1]$ by deleting one element of type 1 and another of either type 1 or type 2. Call these relations $\mathbf{R}_{1,1}$ and $\mathbf{R}_{1,2}$. If $a=1$, there are no $\mathbf{R}_{1,1}$ relations. Occasionally, we use a superscript with $\mathbf{R}_W$ to denote the grading.

If $a>1$ and ψ is a uniform homomorphism, then $\psi(\mathbf{R}_{1,1})$ is

$$\psi(R^{m-1})+(a-2)\psi(Y_1)+b\psi(Y_2)+\tbinom{a-2}{2}\psi(Y_{1,1})+(a-2)b\psi(Y_{1,2})=0. \tag{2.6}$$

These coefficients count the number of relevant subsets $S\subset L$. For example, in the last term, since this L contains $a-2$ numbers of type 1 and b numbers of type 2, there are $(a-2)b$ ways to choose a set $S\subset L$ such that $S\cup\{n\}$ is short and S has one type-1 element and one type-2 element, and ψ sends each of them to the same element of $\mathbb{Z}_2$. Similarly, abbreviating $\psi(Y_W)$ as ψ_W, and $\psi(R)$ as ψ_0, $\psi(\mathbf{R}_{1,2})$ is the relation

$$\psi_0+(a-1)\psi_1+(b-1)\psi_2+\tbinom{a-1}{2}\psi_{1,1}+(a-1)(b-1)\psi_{1,2}=0. \tag{2.7}$$

Proposition 2.8. *Let $\phi: H^m(X)\to\mathbb{Z}_2$ be the Poincaré duality isomorphism, and let $\phi_W=\phi(Y_W)$. Then*

$$\begin{aligned}\phi_{1,1}=\phi_{1,2}&=1\\ \phi_2&=a-1\\ \phi_1&=a+b\\ \phi_0&=(a-1)b+\tbinom{a-1}{2}.\end{aligned}$$

Proof. By symmetry, ϕ is uniform. Using the notation introduced above, there are relations in $H^m(X)$ of the form $\mathbf{R}_1$ and $\mathbf{R}_2$ satisfying, respectively,

$$\begin{aligned}\phi_0+(a-1)\phi_1+b\phi_2+\tbinom{a-1}{2}\phi_{1,1}+(a-1)b\phi_{1,2}&=0\\ \phi_0+a\phi_1+(b-1)\phi_2+\tbinom{a}{2}\phi_{1,1}+a(b-1)\phi_{1,2}&=0,\end{aligned}$$

as well as relations $\mathbf{R}^m_{1,1}$ and $\mathbf{R}^m_{1,2}$ like (2.6) and (2.7), but with ψ replaced by ϕ. As one can check by row-reduction or substitution, the nonzero solution of these four equations (mod 2) is the one stated in the proposition. ∎

Next we expand the expressions in Theorem 2.5. The parity of a is not an issue in these expansions. Part (a) expands, in bidegree $(m, m-1)$, as (in our new notation)

$$\begin{aligned}&\sum_{i=1}^{2m-2-2^e}\tbinom{2m-1-2^e}{i}\tbinom{2^e-1}{m-i-1}w_1^iw_2R^{m-i-1}\otimes w_1^{2m-1-2^e-i}R^{2^e-m+i}\\ &+w_2R^{m-1}\otimes w_1^{2m-1-2^e}R^{2^e-m}+w_1^{2m-1-2^e}w_2R^{2^e-m}\otimes R^{m-1}\\ &+\sum_{i=1}^{2m-2-2^e}\tbinom{2m-1-2^e}{i}\tbinom{2^e-1}{m-i}w_1^iR^{m-i}\otimes w_1^{2m-1-2^e-i}w_2R^{2^e-m+i-1}\\ &+\tbinom{2^e-1}{m}R^m\otimes w_1^{2m-1-2^e}w_2R^{2^e-m-1}+\tbinom{2^e-1}{2^e+1-m}w_1^{2m-1-2^e}R^{2^e+1-m}\otimes w_2R^{m-2}.\end{aligned}$$

The first line is $\sum_{i=1}^{2m-2-2^e}\binom{2m-1-2^e}{i}\binom{2^e-1}{m-i-1}$ times $Y_{1,2}\otimes Y_1$. This sum is easily seen to be 0 mod 2. Similarly the sum on the third line is $\equiv 0$. We obtain that the expansion in part (a) is, in bidegree $(m, m-1)$,

$$Y_2\otimes Y_1+Y_{1,2}\otimes R^{m-1}+(1+\delta_{m,2^e})R^m\otimes Y_{1,2}+Y_1\otimes Y_2. \tag{2.9}$$

We frequently use Lucas's Theorem for evaluation of mod 2 binomial coefficients. For example, here we use that $\binom{2^e-1}{i}\equiv 1$ for all nonnegative $i\le 2^e-1$.

Part (b) expands, in bidegree $(m, m-1)$, as

$$\begin{aligned}&\sum_{i=1}^{m-2}\tbinom{m-1}{i}\tbinom{m-2}{m-i-2}w_1^iw_2^2R^{m-i-2}\otimes w_1^{m-1-i}R^i\\ &+\sum_{i=2}^{m-2}\tbinom{m-1}{i}\tbinom{m-2}{m-i}w_1^iR^{m-i}\otimes w_1^{m-1-i}w_2^2R^{i-2}\\ &+w_2^2R^{m-2}\otimes w_1^{m-1}+mw_1^{m-1}R\times w_2^2R^{m-3}.\end{aligned}$$

For $m\ne 2^{e-1}+1$,

$$\sum_{i=1}^{m-2}\tbinom{m-1}{i}\tbinom{m-2}{m-i-2}\equiv\tbinom{2m-3}{m-2}+1\equiv 1$$

and

$$\sum_{i=1}^{m-2}\tbinom{m-1}{i}\tbinom{m-2}{m-i-2}\equiv\tbinom{2m-3}{m}+m\equiv\delta_{m,2^e}+m,$$

and so, similarly to (2.4), the expansion equals

$$Y_{1,2}\otimes Y_1+(\delta_{m,2^e}+m)Y_1\otimes Y_{1,2}+Y_2\otimes Y_1+mY_1\otimes Y_2. \tag{2.10}$$

The expansion of part (c) of Theorem 2.5 is easier. It equals, in bidegree $(m, m-1)$,

$$w_1w_2^{2^{e-1}} \otimes w_1^{2^{e-1}} + w_1^{2^{e-1}+1} \otimes w_2^{2^{e-1}} = Y_{1,2} \otimes Y_1 + Y_1 \otimes Y_2. \qquad (2.11)$$

Now we show, one-at-a-time, that there are uniform homomorphisms ψ such that $\phi \otimes \psi$ sends (2.9), (2.10), and (2.11) to 1.

If $\psi : H^{m-1}(X) \to \mathbb{Z}_2$ is a uniform homomorphism, applying $\phi \otimes \psi$ to (2.9) with a even yields, using Proposition 2.8,

$$\psi(Y_1) + \psi(Y_0) + \varepsilon_1\psi(Y_{1,2}) + b\psi(Y_2). \qquad (2.12)$$

Here and in the following, ε_t denotes an element of $\mathbb{Z}_2$ whose value turns out to be irrelevant. To have (2.12) be nonzero, we need a uniform homomorphism ψ satisfying the system with the following augmented matrix. The columns represent $\psi(Y_0)$, $\psi(Y_1)$, $\psi(Y_2)$, $\psi(Y_{1,1})$, and $\psi(Y_{1,2})$, respectively, and the second and third rows are (2.6) and (2.7).

$$\left[\begin{array}{ccccc|c} 1 & 1 & b & 0 & \varepsilon_1 & 1 \\ 1 & 0 & b & \varepsilon_2 & 0 & 0 \\ 1 & 1 & b-1 & \varepsilon_2 & b-1 & 0 \end{array}\right]$$

Here we have noted that since a is even, $\binom{a-2}{2} \equiv \binom{a-1}{2} \bmod 2$. This system is easily seen to have a solution, proving part (a) of Theorem 2.5.

Applying $\phi \otimes \psi$ to (2.10) with a odd and $a > 1$ similarly yields

$$\psi(Y_1) + \varepsilon_3\psi(Y_{1,2}) + m(b+1)\psi(Y_2).$$

Now ψ must satisfy the following system, which is also easily seen to have a solution.

$$\left[\begin{array}{ccccc|c} 0 & 1 & m(b+1) & 0 & \varepsilon_3 & 1 \\ 1 & 1 & b & \varepsilon_4 & b & 0 \\ 1 & 0 & b-1 & \varepsilon_4+1 & 0 & 0 \end{array}\right]$$

Here we have used that if a is odd, then $\binom{a-1}{2} \equiv \binom{a-2}{2} + 1$. If $a = 1$, then the fourth column and second row are removed, and again there is a solution.

Finally, applying $\phi \otimes \psi$ to (2.11) with a odd leads to the following system for ψ, which again has a solution.

$$\left[\begin{array}{ccccc|c} 0 & 1 & b+1 & 0 & 0 & 1 \\ 1 & 1 & b & \varepsilon_4 & b & 0 \\ 1 & 0 & b-1 & \varepsilon_4+1 & 0 & 0 \end{array}\right]$$

Again, the fourth column and second row are omitted if $a = 1$, but there is still a solution. This completes the proof of Theorem 2.5.

3. Proof of 1.2 for genetic codes with a single gene of size 4

In this section we prove (1.3) for $X = \overline{M}(\bar{\ell})$ when the genetic code of $\bar{\ell}$ is $\langle\{n, a+b+c, a+b, a\}\rangle$ with $n > a+b+c > a+b > a > 0$. The analysis is similar to that of $\langle\{n, a+b, a\}\rangle$ in the previous section, except that there are many more cases to consider.

Proposition 2.1 easily implies

Proposition 3.1. *For X as above, if $3 \le d \le n-6$, $H^d(X)$ has basis*

$$\{R^d\} \cup \{T_i^d:\ 1 \le i \le a+b+c\} \cup \{T_{i,j}^d:\ 1 \le i \le a+b,\ i < j \le a+b+c\}$$
$$\cup\{T_{i,j,k}^d:\ 1 \le i \le a,\ i < j \le a+b,\ j < k \le a+b+c\}.$$

For $n-5 \le d \le n-3$, these classes span $H^d(X)$ but are subject to additional relations.

We adopt notation similar to that of the preceding proof, using $\theta : [a+b+c] \to [3]$ sending intervals $[1, a]$, $(a, a+b]$, and $(a+b, a+b+c]$ to 1, 2, and 3, respectively. As before, we let $Y_{\theta(S)} = T_S$, $w_{\theta(i)} = V_i$, and $m = n-3$.

We begin by proving, similarly to Proposition 2.8,

Theorem 3.2. *Let $\phi : H^m(X) \to \mathbb{Z}_2$ be the Poincaré duality isomorphism, and let $\phi_W = \phi(Y_W)$. Then*

$$\begin{aligned}
\phi_{1,1,1} = \phi_{1,1,2} = \phi_{1,1,3} &= 1\\
\phi_{1,2,2} = \phi_{1,2,3} &= 1\\
\phi_{2,2} = \phi_{2,3} &= a-1\\
\phi_{1,3} &= a+b\\
\phi_{1,1} = \phi_{1,2} &= a+b+c-1\\
\phi_3 &= (a-1)(b-1) + \tbinom{a}{2}\\
\phi_2 &= (a-1)(b+c) + \tbinom{a}{2}\\
\phi_1 &= (a-1)(a+b+c-1) + \tbinom{a-1}{2} + \tbinom{b}{2} + (b-1)(c-1)\\
\phi_0 &= \tbinom{a}{2}(a+b+c-1) + (a-1)(\tbinom{b}{2} + (b-1)(c-1)).
\end{aligned}$$

Proof. Let $\mathcal{S} =$

$\{(1),(2),(3),(1,1),(1,2),(1,3),(2,2),(2,3),(1,1,1),(1,1,2),(1,1,3),(1,2,2),(1,2,3)\}$

denote the set of types of elements that can be deleted from $[n-1]$ yielding a long subset $L \subset [n-1]$. For example, $(1,2)$ refers to a set $[n-1]-\{x,y\}$ with $x \le a$ and $a < y \le a+b$. For $U \in \mathcal{S}$ and $1 \le i \le 3$, let u_i denote the number of i's in U. For example, if $U=(1,1,2)$, then $u_1 = 2$, $u_2 = 1$, and $u_3 = 0$. For each $U \in \mathcal{S}$, there is a relation $\mathbf{R}_U$ of type (2.2), and $\phi(\mathbf{R}_U)$ is

$$\phi_0 + \sum_{U' \in \mathcal{S}} \tbinom{a-u_1}{u_1'}\tbinom{b-u_2}{u_2'}\tbinom{c-u_3}{u_3'}\phi_{U'} = 0. \tag{3.3}$$

For example, (2.7) is of this form, including only a and b (not c) and corresponding to $U=(1,2)$, and with ψ instead of ϕ. The set of all equations (3.3) is a system of 13 homogeneous equations over $\mathbb{Z}_2$ in 14 unknowns $\phi_{U'}$, and its nonzero solution is the one stated in the theorem.

This solution was found manually by row reduction, and then checked by `Maple`, noting that the system only depends on a mod 4, b mod 4, and c mod 2. The program verified that the solution worked in all 32 cases.

There are special considerations when $a = 1$ or 2, or $b = 1$. For example, if $b = 1$, then $\binom{b-2}{2}$ should be 0 for us, but is 1 in most binomial coefficient formulas. But the relations $\mathbf{R}_U$ in which such coefficients appear are not present, and so the equations (3.3) for these U need not be considered. ■

Let

$$\mathcal{S}' = \{(1,1),(1,2),(1,3),(2,2),(2,3),(1,1,1),(1,1,2),(1,1,3),(1,2,2),(1,2,3)\}. \tag{3.4}$$

For $\psi : H^{m-1}(X) \to \mathbb{Z}_2$ to be a uniform homomorphism with $\psi_W = \psi(Y_W)$, the conditions that must be satisfied are, for $U \in \mathcal{S}'$,

$$\psi_0 + \sum_{U' \in \mathcal{S}} \tbinom{a-u_1}{u_1'}\tbinom{b-u_2}{u_2'}\tbinom{c-u_3}{u_3'}\psi_{U'} = 0. \tag{3.5}$$

The difference between the situation here and in the preceding proof is that relations in $H^m(X)$ allow sets L (in (2.2)) with $|L| = m+1 = n-2$, but relations in $H^{m-1}(X)$ do not.

To prove (1.3), we seek $\psi : H^{m-1}(X) \to \mathbb{Z}_2$ satisfying (3.5) for all $U \in \mathcal{S}'$, and $\{v_1, \ldots, v_{2m-1}\}$ such that

$$(\phi \otimes \psi)(\prod_{i=1}^{2m-1}(Dv_i)) = 1. \tag{3.6}$$

The classes v_i that we will use depend on the mod 4 values of a and b, and c mod 2. First we deal with two cases that turn out to be exceptional. The following easily-verified lemma will be useful.

Lemma 3.7. $\binom{A}{2} + AB + AC + BC + \binom{B}{2} \equiv 0$ *mod 2 iff* $A + B \equiv 0 \mod 4$ *or* $A + B + 2C \equiv 1 \mod 4$.

The following result is our first verification of (3.6).

Proposition 3.8. *Suppose* $a \equiv b \equiv 1 \mod 4$ *and* $c \equiv 1 \mod 2$. *The isomorphism* $\phi : H^m(X) \to \mathbb{Z}_2$ *sends each* $Y_{i,j,k}$ *to* 1 *and other monomials to 0. There exists* $\psi : H^{m-1}(X) \to \mathbb{Z}_2$ *sending each* $Y_{i,j}$ *to 1 and other monomials to 0. Let* $\varepsilon = 1$ *if* $m - 2$ *is a 2-power, and* $\varepsilon = 2$ *otherwise. Then*

$$(\phi \otimes \psi)((Dw_1)^{m-\varepsilon}(Dw_2)^2(Dw_3)^3(DR)^{m-6+\varepsilon}) = 1.$$

Proof. The ϕ-part is easily checked using Theorem 3.2, and the ψ-part follows from Lemma 3.7. Indeed, (3.5) becomes

$\binom{a-u_1}{2} + (a - u_1)(b - u_2) + (a - u_1)(c - u_3) + \binom{b-u_2}{2} + (b - u_2)(c - u_3)$,

which, by 3.7, is 0 for the prescribed a, b, c if $u_1 + u_2 \equiv 2 \mod 4$ or $u_1 + u_2 + 2u_3 \equiv 3$ mod 4, and this is easily verified to be true for the ten elements of $\mathcal{S}'$.

In the expansion of $(Dw_1)^{m-\varepsilon}(Dw_2)^2(Dw_3)^3(DR)^{m-6+\varepsilon}$, there are no terms with repeated subscripts in either Y factor since it only involves one element of each type. Also, there are no $Y_{1,2,3} \otimes Y_{1,2}$ or $Y_{1,2,3} \otimes Y_{2,3}$ terms, since $(Dw_2)^2 = w_2^2 \otimes 1 + 1 \otimes w_2^2$. When $\varepsilon = 2$, the $Y_{1,2,3} \otimes Y_{1,3}$ terms come from

$$\begin{aligned} & \sum_{i=1}^{m-3} \tbinom{m-2}{i}\tbinom{m-4}{m-i-4} w_1^i w_2^2 w_3^2 R^{m-i-4} \otimes w_1^{m-2-i} w_3 R^i \\ + & \sum_{i=1}^{m-3} \tbinom{m-2}{i}\tbinom{m-4}{m-i-3} w_1^i w_2^2 w_3 R^{m-i-3} \otimes w_1^{m-2-i} w_3^2 R^{i-1} \\ = & \ (\tbinom{2m-6}{m-4} + 1) Y_{1,2,3} \otimes Y_{1,3} = Y_{1,2,3} \otimes Y_{1,3} \end{aligned}$$

since $m-2$ is not a 2-power. If $m-2$ is a 2-power, the similar calculation, involving $\sum\binom{m-1}{i}\binom{m-5}{m-i-4}$ and $\sum\binom{m-1}{i}\binom{m-5}{m-i-3}$, gives just $\binom{2m-6}{m-4}=1$. ■

The other exceptional case verifying (3.6) is similar.

Proposition 3.9. *Suppose $a\equiv 2 \mod 4$, $b\equiv 4 \mod 4$, and $c\equiv 1 \mod 2$. The isomorphism $\phi: H^m(X)\to\mathbb{Z}_2$ sends $Y_{i,j,k}$, $Y_{2,2}$, and $Y_{2,3}$ to 1, and other monomials to 0. There exists $\psi: H^{m-1}(X)\to\mathbb{Z}_2$ sending each $Y_{i,j}$ to 1 and other monomials to 0. Let $\varepsilon=1$ if $m-2$ is a 2-power, and $\varepsilon=2$ otherwise. Then*

$$(\phi\otimes\psi)((Dw_1)^2(Dw_2)^2(Dw_3)^{m-\varepsilon}(DR)^{m-5+\varepsilon})=1.$$

Proof. The only term in the expansion which is mapped nontrivially is $Y_{2,3}\otimes Y_{1,3}$. It appears as

$$\sum_{i=1}^{m-\varepsilon-1}\binom{m-\varepsilon}{i}\binom{m-5+\varepsilon}{m-2-i}w_2^2w_3^iR^{m-2-i}\otimes w_1^2w_3^{m-\varepsilon-i}R^{i+\varepsilon-3}$$

with coefficient $\binom{2m-5}{m-2}+\binom{m-5+\varepsilon}{m-2}+\binom{m-5+\varepsilon}{\varepsilon-2}=1$. ■

Let $\overline{a}$ (resp. $\overline{b}$) denote the mod 4 value of a (resp. b), and $\overline{c}$ the mod 2 value of c. For the other 30 cases of $\overline{a}$, $\overline{b}$, and $\overline{c}$ (or 90 if you consider deviations regarding whether m or $m-1$ is a 2-power), we use `Maple` to tell that an appropriate ψ can be found. To accomplish this in all cases, several choices for the exponents of Dw_1, Dw_2, and Dw_3 are required. Possibly some choice of exponents might work in all cases, but we did not find one.

Most of our results will be obtained using the following result.

Proposition 3.10. *If neither m nor $m-1$ is a 2-power, then the component of $(Dw_1)^\alpha(Dw_2)^2(Dw_3)(DR)^{2m-4-\alpha}$ in bidegree $(m, m-1)$ equals*

$$\begin{aligned}
& \tbinom{2m-4-\alpha}{m}(Y_0 \otimes Y_{1,2,3} + Y_1 \otimes Y_{1,2,3}) \\
+\ & \tbinom{2m-4-\alpha}{m-1}(Y_{1,2,3} \otimes Y_0 + Y_{1,2,3} \otimes Y_1 + Y_3 \otimes Y_{1,2} + Y_{1,3} \otimes Y_{1,2}) \\
+\ & \tbinom{2m-4-\alpha}{m-2}(Y_2 \otimes Y_{1,3} + Y_{1,2} \otimes Y_3) \\
+\ & \tbinom{2m-4-\alpha}{m-3}(Y_{2,3} \otimes Y_1 + Y_{1,2,3} \otimes Y_1 + Y_{1,3} \otimes Y_2 + Y_{1,3} \otimes Y_{1,2}) \\
+\ & \tbinom{2m-4-\alpha}{m-4}(Y_1 \otimes Y_{2,3} + Y_1 \otimes Y_{1,2,3}).
\end{aligned}$$

If m is a 2-power, there is an additional term $Y_1 \otimes Y_{1,2,3}$. If $m-1$ is a 2-power, $Y_{1,2,3} \otimes Y_1 + Y_{1,3} \otimes Y_{1,2}$ must be added to the above expansion.

Proof. The desired expression expands as

$$\begin{aligned}
& \sum_{i=0}^{\alpha} \tbinom{\alpha}{i}\tbinom{2m-4-\alpha}{m-i-3} w_1^i w_2^2 w_3 R^{m-i-3} \otimes w_1^{\alpha-i} R^{m-1-\alpha+i} \\
+\ & \sum_{i=0}^{\alpha} \tbinom{\alpha}{i}\tbinom{2m-4-\alpha}{m-i} w_1^i R^{m-i} \otimes w_1^{\alpha-i} w_2^2 w_3 R^{m-4-\alpha-i} \\
+\ & \sum_{i=0}^{\alpha} \tbinom{\alpha}{i}\tbinom{2m-4-\alpha}{m-i-2} w_1^i w_2^2 R^{m-i-2} \otimes w_1^{\alpha-i} w_3 R^{m-2-\alpha+i} \\
+\ & \sum_{i=0}^{\alpha} \tbinom{\alpha}{i}\tbinom{2m-4-\alpha}{m-i-1} w_1^i w_3 R^{m-i-1} \otimes w_1^{\alpha-i} w_2^2 R^{m-3-\alpha+i}.
\end{aligned}$$

If neither m nor $m-1$ is a 2-power, the coefficients $\binom{2m-4}{m-t}$ for $t = 3, 0, 2, 1$, which occur as the sum of all coefficients on a line, are 0. Thus the four lines equal, respectively,

$$\begin{aligned}
& \tbinom{2m-4-\alpha}{m-3}(Y_{2,3} \otimes Y_1 + Y_{1,2,3} \otimes Y_1) + \tbinom{2m-4-\alpha}{m-\alpha-3}(Y_{1,2,3} \otimes Y_0 + Y_{1,2,3} \otimes Y_1), \\
& \tbinom{2m-4-\alpha}{m}(Y_0 \otimes Y_{1,2,3} + Y_1 \otimes Y_{1,2,3}) + \tbinom{2m-4-\alpha}{m-\alpha}(Y_1 \otimes Y_{2,3} + Y_1 \otimes Y_{1,2,3}), \\
& \tbinom{2m-4-\alpha}{m-2}(Y_2 \otimes Y_{1,3} + Y_{1,2} \otimes Y_{1,3}) + \tbinom{2m-4-\alpha}{m-\alpha-2}(Y_{1,2} \otimes Y_3 + Y_{1,2} \otimes Y_{1,3}), \\
& \tbinom{2m-4-\alpha}{m-1}(Y_3 \otimes Y_{1,2} + Y_{1,3} \otimes Y_{1,2}) + \tbinom{2m-4-\alpha}{m-1-\alpha}(Y_{1,3} \otimes Y_2 + Y_{1,3} \otimes Y_{1,2}).
\end{aligned}$$

The sum of these is easily manipulated into the claimed form. If m is a 2-power, then $\binom{2m-4}{m}$ is odd, while if $m-1$ is a 2-power, $\binom{2m-4}{m-1}$ and $\binom{2m-4}{m-3}$ are odd, yielding the additional terms in the claim. ■

The following result follows immediately from Proposition 3.10 and Theorem 3.2.

Corollary 3.11. *Let $q_t=\binom{2m-4-\alpha}{m-t}$ for $0\le t\le 4$, and $\psi_W=\psi(Y_W)$, where $\psi: H^{m-1}(X)\to\mathbb{Z}_2$ is a uniform homomorphism. Then*

$$\begin{aligned}
&(\phi\otimes\psi)((Dw_1)^\alpha(Dw_2)^2(Dw_3)(DR)^{2m-4-\alpha}) \qquad (3.12)\\
=\;& q_1\psi_0+(q_1+q_3a)\psi_1+q_3(a+b)\psi_2+q_2(a+b+c-1)\psi_3\\
&+(q_1(ab+1+\tbinom{a}{2}))+q_3(a+b))\psi_{1,2}+q_2((a-1)(b+c)+\tbinom{a}{2}))\psi_{1,3}\\
&+q_4((a-1)(a+b+c-1)+\tbinom{a-1}{2}+\tbinom{b}{2}+(b-1)(c-1))(\psi_{2,3}+\psi_{1,2,3})\\
&+q_0((a+\tbinom{a}{2}-1)(a+b+c-1)+\tbinom{a-1}{2}+a\tbinom{b}{2}+a(b-1)(c-1))\psi_{1,2,3}.
\end{aligned}$$

Lemma 3.13. *If $m=2^e+m'$ with $2\le m'\le 2^e-1$ and $\alpha=2m'-3$ and $0\le t\le 4$, then $\binom{2m-4-\alpha}{m-t}\equiv 1$ mod 2.*

Proof. The top of the binomial coefficient is $2^{e+1}-1$, while the bottom is $\le 2^{e+1}-1$. ■

The first of several verifications of (1.3) for multiple values of $(\overline{a},\overline{b},\overline{c})$ appears in the following result.

Theorem 3.14. *If $m=2^e+m'$ with $2\le m'\le 2^e-1$, then*

$$(Dw_1)^{2m'-3}(Dw_2)^2(Dw_3)(DR)^{2^{e+1}-1}\ne 0\in H^{2m-1}(X\times X)$$

for the values of $\overline{a}$, $\overline{b}$, and $\overline{c}$ which have an $\times$ in the 3.14 column of Table 3.23.

Proof. This is the case described in Lemma 3.13, so that $q_0=\cdots=q_4=1$ in (3.12). We need values of ψ_W such that the RHS of (3.12) equals 1, and (3.5) is satisfied for all $U\in\mathcal{S}'$. (Recall that the relationship of U to (3.5) is that u_i is the number of occurrences of i in U.) Altogether this is 11 equations over $\mathbb{Z}_2$ in 14 unknowns. The coefficients of the equations depend only on $\overline{a}$, $\overline{b}$, and $\overline{c}$. It is a simple matter to run `Maple` on these 32 cases, and it tells us that there is a solution in exactly the claimed cases. The `Maple` program, input and output, can be seen at [2]. The two cases, $(\overline{a},\overline{b},\overline{c})=(1,1,1)$ and $(2,4,1)$, considered in Propositions 3.8 and 3.9 are not included in Table 3.23 because they did not yield a solution in any of the situations whose results appear as a column of that table. The special situation when $a=1$ or 2 or $b=1$ is not a problem, exactly as in the proof of Theorem 3.2. ■

The next result is very similar. The only difference is a small change in the exponent of Dw_1 (and hence also of DR). This changes the values of q_t.

Theorem 3.15. *If $m = 2^e + m'$ with $2 \le m' \le 2^e - 1$, then*

$$(Dw_1)^{2m'-2}(Dw_2)^2(Dw_3)(DR)^{2^{e+1}-2} \neq 0 \in H^{2m-1}(X \times X)$$

for the values of $\overline{a}$, $\overline{b}$, and $\overline{c}$ which have an $\times$ in the 3.15 column of Table 3.23.

Proof. In this case, $q_t = m - t - 1$. That is the only change from the proof of Theorem 3.14. Here we require that solution must exist both when $\overline{q} = (1, 0, 1, 0, 1)$ and $(0, 1, 0, 1, 0)$, covering either parity of m. Here and later $\overline{q} = (q_0, q_1, q_2, q_3, q_4)$. ■

The third result also just involves a change in the exponent of Dw_1. This time $q_t = \binom{m-t+2}{2}$, so we require a solution for all four of the vectors $\overline{q}$, corresponding to mod 4 values of m.

Theorem 3.16. *If $m = 2^e + m'$ with $2 \le m' \le 2^e - 1$, then*

$$(Dw_1)^{2m'-1}(Dw_2)^2(Dw_3)(DR)^{2^{e+1}-3} \neq 0 \in H^{2m-1}(X \times X)$$

for the values of $\overline{a}$, $\overline{b}$, and $\overline{c}$ which have an $\times$ in the 3.16 column of Table 3.23.

We can fill in the missing cases by changing the exponent of Dw_2. We begin with the following analogue of Proposition 3.10, whose proof is totally analogous.

Proposition 3.17. *If neither m nor $m - 1$ is a 2-power, then the component of $(Dw_1)^\alpha(Dw_2)(Dw_3)(DR)^{2m-3-\alpha}$ in bidegree $(m, m-1)$ equals*

$$\begin{aligned} & \tbinom{2m-3-\alpha}{m}(Y_0 \otimes Y_{1,2,3} + Y_1 \otimes Y_{1,2,3}) \\ + \quad & \tbinom{2m-3-\alpha}{m-1}(Y_{1,2,3} \otimes Y_0 + Y_{1,2,3} \otimes Y_1 + Y_3 \otimes Y_{1,2} + Y_{1,3} \otimes Y_{1,2} + Y_2 \otimes Y_{1,3} + Y_{1,2} \otimes Y_{1,3}) \\ + \quad & \tbinom{2m-3-\alpha}{m-2}(Y_{2,3} \otimes Y_1 + Y_{1,2,3} \otimes Y_1 + Y_{1,3} \otimes Y_2 + Y_{1,3} \otimes Y_{1,2} + Y_{1,2} \otimes Y_3 + Y_{1,2} \otimes Y_{1,3}) \\ + \quad & \tbinom{2m-3-\alpha}{m-3}(Y_1 \otimes Y_{2,3} + Y_1 \otimes Y_{1,2,3}). \end{aligned}$$

If m is a 2-power, there is an additional $Y_1 \otimes Y_{1,2,3}$.

We do not need to use this proposition when $m - 1$ is a 2-power.

Theorem 3.18. *If $m = 2^e + m'$ with $2 \le m' \le 2^e - 1$, then*

$$(Dw_1)^{2m'-1}(Dw_2)(Dw_3)(DR)^{2^{e+1}-2} \ne 0 \in H^{2m-1}(X \times X)$$

for the values of $\overline{a}$, $\overline{b}$, and $\overline{c}$ which have an $\times$ in the 3.18 column of Table 3.23.

Proof. Let $q'_t = \binom{2m-3-\alpha}{m-t}$ for $0 \le t \le 3$. Using Proposition 3.17 and Theorem 3.2, we find that

$$\begin{aligned}
&(\phi \otimes \psi)((Dw_1)^\alpha (Dw_2)(Dw_3)(DR)^{2m-3-\alpha}) \qquad (3.19)\\
= \;& q'_1\psi_0 + (q'_1 + q'_2 a)\psi_1 + q'_2(a+b)\psi_2 + q'_2(a+b+c-1)\psi_3\\
&+(q'_1(ab+1+\tbinom{a}{2})) + q'_2(a+b))\psi_{1,2}\\
&+(q'_1(a(b+c)+a-1+\tbinom{a}{2})) + q'_2(a+b+c-1))\psi_{1,3}\\
&+q'_3((a-1)(a+b+c-1)+\tbinom{a-1}{2}+\tbinom{b}{2}+(b-1)(c-1))(\psi_{2,3}+\psi_{1,2,3})\\
&+q'_0((a+\tbinom{a}{2}-1)(a+b+c-1)+\tbinom{a-1}{2}+a\tbinom{b}{2}+a(b-1)(c-1))\psi_{1,2,3}.
\end{aligned}$$

Similarly to Lemma 3.13, with $\alpha = 2m' - 1$, we have $\binom{2m-3-\alpha}{m-t} \equiv m - t - 1 \mod 2$, and the result follows similarly to the three previous ones, having `Maple` check whether there is a solution to the system of 11 equations in 14 unknowns, whose first equation is that the RHS of (3.19) equals 1 and others are, as before, (3.5) for each $U \in \mathcal{S}'$. This time a solution is required for both $\overline{q}' = (0, 1, 0, 1)$ and $(1, 0, 1, 0)$. ■

Referring to Table 3.23 and Theorems 3.14, 3.15, 3.16, and 3.18, accompanied by Propositions 3.8 and 3.9, we find that, when neither m nor $m - 1$ is a 2-power, (1.3) is satisfied for all $(\overline{a}, \overline{b}, \overline{c})$, establishing Theorem 1.2 when neither m nor $m - 1$ is a 2-power. Next we handle the case when $m = 2^e$.

Theorem 3.20. *If $m = 2^e$, then, for $1 \le \varepsilon \le 3$,*

$$(Dw_1)^{2^e-\varepsilon}(Dw_2)^2(Dw_3)(DR)^{2^e+\varepsilon-4} \ne 0 \in H^{2m-1}(X \times X)$$

for the values of $\overline{a}$, $\overline{b}$, and $\overline{c}$ which have an $\times$ in the 3.20(ε) column of Table 3.23.

Proof. Because of the change due to $m = 2^e$ noted in Proposition 3.10, the expression in Corollary 3.11 has an extra $((a-1)(a+b+c-1)+\binom{a-1}{2}+\binom{b}{2}+(b-1)(c-1))\psi_{1,2,3}$ added. The vectors $\overline{q}$ are $(0, 1, 1, 1, 1)$, $(0, 0, 1, 0, 1)$, and $(0, 0, 0, 1, 1)$ for $\varepsilon = 3, 2, 1$, respectively. The other 10 equations for the ψ_W's are as before. `Maple` tells us when the system has a solution. ■

The next result is the 2^e-analogue of Theorem 3.18. As shown in Table 3.23, this, Theorem 3.20, and Propositions 3.8 and 3.9 imply Theorem 1.2 when $m = 2^e$.

Theorem 3.21. *If $m = 2^e$, then*

$$(Dw_1)^{2^e-1}(Dw_2)(Dw_3)(DR)^{2^e-2} \neq 0 \in H^{2m-1}(X \times X)$$

for the values of $\overline{a}$, $\overline{b}$, and $\overline{c}$ which have an $\times$ in the 3.21 column of Table 3.23.

Proof. Because of the change due to $m = 2^e$ noted in Proposition 3.17, the expression in (3.19) has an extra $((a-1)(a+b+c-1)+\binom{a-1}{2}+\binom{b}{2}+(b-1)(c-1))\psi_{1,2,3}$ added. The vector $\overline{q}'$ is $(0,0,1,0)$, and the other 10 equations for the ψ_W's are as before. `Maple` tells us when the system has a solution. ■

Finally, we handle the case $m = 2^e + 1$.

Theorem 3.22. *If $m = 2^e + 1$, then, for $\varepsilon = \pm 1$,*

$$(Dw_1)^{2^e+\varepsilon}(Dw_2)^2(Dw_3)(DR)^{2^e-\varepsilon-2} \neq 0 \in H^{2m-1}(X \times X)$$

for the values of $\overline{a}$, $\overline{b}$, and $\overline{c}$ which have an $\times$ in the 3.22(ε) column of Table 3.23.

Proof. Because of the change due to $m = 2^e + 1$ noted in Proposition 3.10, the expression in Corollary 3.11 has an extra $\psi_1 + (a+b)\psi_{1,2}$ added. The vectors $\overline{q}$ are $(0,0,0,0,1)$ and $(0,0,1,1,1)$ for $\varepsilon = 1$ and -1, respectively. The other 10 equations for the ψ_W's are as before. `Maple` tells us when the system has a solution. ■

Table 3.23.

			$m \neq 2^e, 2^e+1$				$m = 2^e$				$m = 2^e+1$	
$\overline{a}$	$\overline{b}$	$\overline{c}$	3.14	3.15	3.16	3.18	3.20(3)	3.20(2)	3.20(1)	3.21	3.22(1)	3.22(−1)
1	1	2			×				×		×	
1	2	1				×				×	×	×
1	2	2			×	×			×	×	×	
1	3	1		×	×	×		×	×	×	×	×
1	3	2		×	×	×		×	×	×	×	×
1	4	1		×	×	×		×	×	×	×	
1	4	2		×	×	×		×	×	×	×	×
2	1	1	×				×	×	×	×	×	×
2	1	2	×				×		×	×	×	
2	2	1		×			×	×	×	×	×	×
2	2	2			×	×	×	×	×	×	×	×
2	3	1	×				×				×	×
2	3	2	×				×	×	×	×	×	×
2	4	2		×		×		×		×	×	
3	1	1		×	×	×		×	×	×	×	×
3	1	2			×	×	×	×	×	×	×	
3	2	1	×	×	×	×	×		×	×	×	×
3	2	2	×	×		×	×	×		×	×	×
3	3	1		×				×				×
3	3	2		×	×	×	×	×	×	×	×	×
3	4	1	×	×		×	×	×	×	×	×	×
3	4	2	×	×		×	×	×	×	×	×	×
4	1	1	×	×	×	×	×	×	×	×	×	×
4	1	2	×		×	×	×		×	×	×	×
4	2	1	×	×	×	×	×	×				×
4	2	2	×	×		×	×	×		×	×	×
4	3	1	×	×	×	×	×	×	×	×	×	×
4	3	2	×		×	×	×	×	×	×	×	×
4	4	1	×	×	×	×	×	×	×	×	×	×
4	4	2	×	×	×	×	×	×	×	×	×	×

The Maple program that performed all these verifications can be viewed at [2]. We conclude that

Theorem 3.24. *If* $X = \overline{M}(\bar{\ell})$ *with genetic code* $\langle\{n, a+b+c, a+b, a\}\rangle$*, then (3.6) holds for some set of* $2m-1$ *classes* v_i*.*

Proof. Table 3.23 shows that for all $(\overline{a}, \overline{b}, \overline{c})$ except $(1,1,1)$ and $(2,4,1)$ one of the tabulated theorems applies, while the two exceptional cases are covered in Propositions 3.8 and 3.9. ■

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