FOR WHICH 2-ADIC INTEGERS x CAN $\sum\limits_k \binom{x}{k}^{-1}$ BE DEFINED?

DONALD M. DAVIS

Abstract. Let $f(n) = \sum_k \binom{n}{k}^{-1}$. In a previous paper, we defined for a p-adic integer x that $f(x)$ is p-*definable* if $\lim f(x_j)$ exists in $\mathbb{Q}_p$, where x_j denotes the mod p^j reduction of x. We proved that if p is odd, then -1 is the only element of $\mathbb{Z}_p - \mathbb{N}$ for which $f(x)$ is p-definable. For $p = 2$, we proved that if the 1's in the binary expansion of x are eventually extraordinarily sparse, then $f(x)$ is 2-definable. Here we present some conjectures that $f(x)$ is 2-definable for many more 2-adic integers. We discuss the extent to which we can prove these conjectures.

1. Statement of conjectures and their consequences

Let $\mathbb{N} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$ denote the natural numbers (including 0), p-adic integers, and p-adic numbers, respectively, with metric $d_p(x, y) = p^{-\nu_p(x-y)}$. Here and throughout, $\nu_p(-)$ denotes the exponent of p in a rational number. Let $f : \mathbb{N} \to \mathbb{Q}_p$ be defined by

$$f(n) = \sum_{k=0}^{n} \binom{n}{k}^{-1}.$$

In [1], we made the following definition.

Definition 1.1. *Let $x \in \mathbb{Z}_p$, and let x_j denote the mod p^j reduction of x. Then $f(x)$ is p-definable if $\langle f(x_j)\rangle$ is a Cauchy sequence in $\mathbb{Q}_p$.*

Then $f(x)$ could be defined to be the limit in $\mathbb{Q}_p$ of this Cauchy sequence.

We proved in [1] that if p is an odd prime, then $f(x)$ is p-definable if and only if $x = -1$ or $x \in \mathbb{N}$. (Actually, p was required to satisfy a technical condition which is satisfied by all primes less than 10^8, and for which there are no primes which are known not to satisfy it.) We also proved that if $x = \sum 2^{e_i}$ with $e_i < e_{i+1}$, then $f(x)$

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is 2-definable if, roughly, $i+1>2^i$ for all sufficiently large i. The 1's in the binary expansion of such an x are eventually extraordinarily sparse. Here we discuss our attempts to prove that $f(x)$ is 2-definable for many more 2-adic integers.

Let $\alpha(n)$ denote the number of 1's in the binary expansion of n, $\lg(-)=[\log_2(-)]$, and $\nu(-)=\nu_2(-)$. Our strongest conjecture is

Conjecture 1.2. *If $0\le k<2^e$, then*

$$\nu(f(2^e+k)-f(k))\ge e-2\alpha(k)-2.$$

Conjecture 1.2 has been verified for $e\le 15$. In this range, equality holds iff $k=2^e-4$ or 2^e-2. The following result describes the consequence of this conjecture for 2-definability.

Proposition 1.3. *Assume Conjecture 1.2. If the number of 0's minus the number of 1's in x_j approaches ∞ as j goes to ∞, then $f(x)$ is 2-definable.*

We include leading 0's in x_j here, since they will eventually be seen. An alternative statement is that $f(x)$ would be 2-definable if the fraction of 0's in x is greater than 1/2.

Proof of Proposition 1.3. Let $x=\sum_{i=1}^{\infty}2^{e_i}$ with $e_i<e_{i+1}$. The ith distinct point in the sequence of $f(x_j)$'s is $f(2^{e_i}+x_{e_i})$, and the $(i-1)$st distinct point is $f(x_{e_i})$. The distance between these points is 2^{-v}, where

$$v=\nu(f(2^{e_i}+x_{e_i})-f(x_{e_i}))\ge e_i-2\alpha(x_{e_i})-2,$$

according to Conjecture 1.2. The number of 0's in x_{e_i} equals $e_i-\alpha(x_{e_i})$. Our hypothesis says that $e_i-2\alpha(x_{e_i})$ becomes arbitrarily large, and hence the distance between the ith and $(i-1)$st distinct points in the sequence is 2^{-v} where v becomes arbitrarily large. Thus our sequence is Cauchy. □

Although we have very strong evidence for Conjecture 1.2, we feel that we are more likely to be able to prove the following conjecture.

Conjecture 1.4. *If $0\le k<2^{e-1}$, then*

$$\nu(f(2^e+2k+1)-f(2k+1))\ge e-2\lg(k+3)+2\nu(k+1).$$

Conjecture 1.4 has been verified for $e \le 15$. In this range, equality holds iff $k = 2^{e-1} - 2$. The following result describes the consequence of this conjecture for 2-definability.

Proposition 1.5. *Assume Conjecture 1.4. Suppose $x = \sum 2^{e_i}$ has $e_1 = 0$ and $e_i < e_{i+1}$ and satisfies $\lim\limits_{i\to\infty}(e_{i+1} - 2e_i) = \infty$. Then $f(x)$ is 2-definable.*

Note that this would be exponentially stronger than the result proved in [1] and referenced above, but still much weaker than the conclusion of Proposition 1.3.

Proof of Proposition 1.5. Arguing similarly to the previous proof, the distance between consecutive points in the sequence is 2^{-v} with

$$v = \nu(f(2^{e_i} + x_{e_i}) - f(x_{e_i})) \ge e_i - 2\lg(x_{e_i}) - 2 = e_i - 2e_{i-1} - 2$$

according to Conjecture 1.4. Since our assumption is that v becomes arbitrarily large, the sequence is Cauchy. $\square$

2. Steps toward a proof of Conjecture 1.4

In this section, we outline a program which we hope might lead to a proof of Conjecture 1.4. Using symmetry of binomial coefficients, the following result is immediate.

Proposition 2.1. *Let $0 \le k < 2^{e-1}$. If the following two statements are true, then so is Conjecture 1.4.*

i. $\nu\Big(\sum\limits_{i=0}^{k}\big(\binom{2^e+2k+1}{i}^{-1} - \binom{2k+1}{i}^{-1}\big)\Big) \ge e - 2\lg(k+2) + 2\nu(k+1)$,

ii. $\nu\Big(\sum\limits_{i=k+1}^{2^{e-1}+k}\binom{2^e+2k+1}{i}^{-1}\Big) \ge e - 2\lg(k+3) + 2\nu(k+1) - 1.$

Our main result is

Theorem 2.2. *Let $0 \le k < 2^{e-1}$. Then statement i. of Proposition 2.1 is true. Indeed, with*

$$T_i := \binom{2^e+2k+1}{i}^{-1} - \binom{2k+1}{i}^{-1},$$

we have

a. *if* $0 \le i \le [(k-1)/2]$, *then*
$$\nu(T_{2i} + T_{2i+1}) \ge e - 2\lg(k+1) + 2\nu(k+1), \quad \textit{and}$$
b. *if* k *is even, then*
$$\nu(T_k) \ge e - 2\lg(k+2).$$

Our proof will use the standard results that $\nu\binom{m+n}{m} = \alpha(m) + \alpha(n) - \alpha(m+n)$, and that $\nu\binom{m+n}{m}$ equals the number of carries when m and n are added in binary arithmetic. It follows from this that

$$\nu\tbinom{k}{i} \le \lg(k+1) - \nu(k+1), \tag{2.3}$$

since, if $\nu(k+1) = t$, then there cannot be any carries in the last t positions in the binary addition of i and $k-i$.

Proof of part b of Theorem 2.2. We first note that

$$\tbinom{2^e+a}{b}^{-1} - \tbinom{a}{b}^{-1} = -\tbinom{2^e+a}{b}^{-1} \sum_{j\ge 1} 2^{je} \sigma_j(\tfrac{1}{a}, \ldots, \tfrac{1}{a-b+1}), \tag{2.4}$$

where $\sigma_j(-)$ denotes an elementary symmetric function.

Let $k = 2\ell$. Including only the $(j=1)$-term, which we will justify, (2.4) yields that $T_{2\ell}$ has the same 2-exponent as

$$2^e \tbinom{2^e+4\ell+1}{2\ell}^{-1} \left(\tfrac{1}{2\ell+2} + \cdots + \tfrac{1}{4\ell+1}\right). \tag{2.5}$$

Note that $2\ell + 2 \le 2^t \le 4\ell + 1$ iff $2^{t-2} \le \ell \le 2^{t-1} - 1$, and so $\nu(\frac{1}{2\ell+2} + \cdots + \frac{1}{4\ell+1}) = -\lg(\ell) - 2$. Thus the 2-exponent of (2.5) equals $e - \alpha(\ell) - \lg(\ell) - 2 \ge e - 2\lg(2\ell+2)$, as claimed. Here we use that $2\lg(\ell+1) \ge \alpha(\ell) + \lg(\ell)$, which is proved by considering separately $2^t \le \ell < 2^{t+1} - 1$ and $\ell = 2^{t+1} - 1$.

Now we justify including only the term with $j = 1$ in the above sum. Let

$$v_j = \nu(2^{je}\sigma_j(\tfrac{1}{2\ell+2}, \ldots, \tfrac{1}{4\ell+1})).$$

If $\nu(\sigma_1(-)) = -t$, then $v_1 = e - t > 0$, and if $j > 1$ then $v_j > j(e-t) > v_1$, since $\sigma_j(-)$ is a sum of products of j factors, each with 2-exponent $\ge -t$, and at most one equal to $-t$. $\square$

Proof of part a of Theorem 2.2. Including only the $(j=1)$-term of (2.4), which again will be justified, we obtain that $T_{2i}+T_{2i+1}$ equals

(2.6)
$$-2^e\binom{2^e+2k+1}{2i}^{-1}\Bigl(\bigl(\tfrac{1}{2k+1}+\cdots+\tfrac{1}{2k-2i+2}\bigr)\bigl(1+\tfrac{2i+1}{2^e+2k-2i+1}\bigr)+\tfrac{2i+1}{(2^e+2k-2i+1)(2k-2i+1)}\Bigr).$$

Thus, using (2.3) at the second step,

$$\begin{aligned}\nu(T_{2i}+T_{2i+1}) &\geq e-\nu\tbinom{k}{i}+\min(-\lg(2k)+\nu(2^e+2k+2),0)\\ &\geq \min(e+2\nu(k+1)-\lg(k+1)-\lg(k),e-\lg(k+1)+\nu(k+1)),\end{aligned}$$

which is as claimed.

We complete the proof by showing that if $j>1$, then using the j-term of the sum in (2.4) in $T_{2i}+T_{2i+1}$ would give an expression with 2-exponent at least as large as was obtained with $j=1$. Analogous to part of (2.6), the j-term would be, up to odd multiples,

$$2^{je}((2^e+2k+2)\sigma_j(-)+\sigma_{j-1}(-)). \tag{2.7}$$

If $\nu(\sigma_1(-))=-t$, then $\nu(\sigma_j(-))>-jt$. When $k<2^{e-1}-1$, since $e>t$ and $e>\nu(2k+2)$, the claim follows from

$$je+\nu(2k+2)-jt>e+\nu(2k+2)-t$$

and

$$je-(j-1)t>e+\nu(2k+2)-t.$$

If $k=2^{e-1}-1$, then $t=e-1$ and (2.7) has 2-exponent e if $j=1$ (from $\sigma_0(-)$) and a larger value if $j>1$. □

Despite much effort, we have been unable to prove statement ii. of Proposition 2.1. Note that the application to 2-definability given in Proposition 1.5 would be true even if Conjecture 1.4 or Proposition 2.1 did not contain the "$+2\nu(k+1)$."

References

[1] D. M. Davis, *For which p-adic integers x can* $\sum_k \binom{x}{k}^{-1}$ *be defined?*, to appear in Journal of Combinatorics and Number Theory. `http://www.lehigh.edu/~dmd1/define3.pdf`

Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA
E-mail address: dmd1@lehigh.edu