THE SYMMETRIC TOPOLOGICAL COMPLEXITY OF THE CIRCLE

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Abstract. We determine the symmetric topological complexity of the circle, using primarily just general topology.

1. Introduction

Let PX denote the space of all paths in a topological space X, and define $p : PX \to X \times X$ by $p(\sigma) = (\sigma(0), \sigma(1))$. If $V \subset X \times X$, a section $s : V \to PX$ is called a motion planning rule on V. The reduced topological complexity of X, $\mathrm{TC}(X)$, is 1 less than the minimal number of open sets V covering $X \times X$ which admit motion planning rules. The notion of topological complexity was introduced by Farber in [2] in unreduced form, but most recent papers have preferred the reduced notation. Topological complexity can be applied to robotics when X is the space of configurations of a robot.

A set $V \subset X \times X$ is *symmetric* if $(x, y) \in V$ iff $(y, x) \in V$. A symmetric motion planning rule on such a set V is one which satisfies $s(x_1, x_0) = \overline{s(x_0, x_1)}$. Here $\overline{\sigma}(t) = \sigma(1 - t)$.

In [1], (reduced) symmetric topological complexity $\mathrm{TC}^{\Sigma}(X)$ of X was defined to be 1 less than the minimal number of symmetric open sets covering $X \times X$ which admit symmetric motion planning rules. We will prove the following new result.

Theorem 1.1. $\mathrm{TC}^{\Sigma}(S^1) = 2$.

An earlier variant of symmetric topological complexity, $\mathrm{TC}^S(X)$, was introduced in [3]. Employing here the reduced TC terminology, $\mathrm{TC}^S(X)$ equals the minimal number of symmetric open sets covering $X \times X - \Delta$ admitting symmetric motion planning

Date: February 13, 2017.
Key words and phrases. Topological complexity.
2000 *Mathematics Subject Classification*: 55M30, 55N25, 57M20.

rules. Here $\Delta = \{(x,x) \in X \times X\}$ is the diagonal. This notion assumes that one motion planning rule chooses the constant path from x to x, possibly extended over a small neighborhood of Δ, and then considers separately symmetric paths between distinct points. The reduced version employed here has the -1 in the reduction which cancels the $+1$ from the section over the diagonal. As noted in [1, Prop 4.2], it is immediate that for all X

$$\mathrm{TC}^S(X) - 1 \le \mathrm{TC}^\Sigma(X) \le \mathrm{TC}^S(X).$$

The advantage of the $\mathrm{TC}^S(-)$ concept is that the space of paths between distinct points of X fibers $\mathbb{Z}_2$-equivariantly over $X \times X - \Delta$, but there is no such fibration over $X \times X$. This leads to a nice cohomological lower bound for $\mathrm{TC}^S(-)$, which we do not have for $\mathrm{TC}^\Sigma(-)$. In an email to the author, Michael Farber confirmed that he felt that the TC^Σ definition was "more natural" than TC^S. One nice feature of TC^Σ is that it is a homotopy invariant ([1, Prop 4.7]), whereas TC^S is not.

In [3], it was shown that $\mathrm{TC}^S(S^n) = 2$ for all $n \ge 1$. Since cohomology shows that when n is even, three (not necessarily symmetric) motion planning rules are required for S^n, we obtain that $\mathrm{TC}^\Sigma(S^n) = 2$ when n is even. In [1, Expl 4.5] and in [5, Expl 17.5], it was noted that for odd n, it was not known whether $\mathrm{TC}^\Sigma(S^n) = 1$ or 2, and the S^1-case was given special attention as an "Open Problem" in [5, 17.6]. Our contribution here is to resolve this open problem.

2. Our approach and an example

Our approach is to associate to a motion planning rule on an open subset of $S^1 \times S^1$ a locally constant function d on an open subset of $I \times I$ with certain properties, and then show (in the next section) that the domains of two such functions cannot cover $I \times I$.

Let $\rho : I \times I \to S^1 \times S^1$ be the usual quotient map defined by $\rho(t,t') = (e^{2\pi i t}, e^{2\pi i t'})$, and $e : \mathbb{R} \to S^1$ the usual covering map defined by $e(t) = e^{2\pi i t}$.

Proposition 2.1. *If $V \subset S^1 \times S^1$ is a symmetric open set, and $s : V \to PS^1$ is a symmetric motion planning rule, there is a continuous function $d : \rho^{-1}(V) \to \mathbb{Z}$*

satisfying, for all points in its domain,

$$d(t,1) - d(t,0) = -1, \tag{2.2}$$

$$d(1,t) - d(0,t) = 1, \tag{2.3}$$

$$\text{and} \quad d(t',t) = -d(t,t'). \tag{2.4}$$

Proof. Suppose $\rho(t,t') \in V$ with $\sigma = s(\rho(t,t')) \in PS^1$. Let $\widetilde{\sigma} : I \to \mathbb{R}$ satisfy $e \circ \widetilde{\sigma} = \sigma$. Note that $\widetilde{\sigma}(1) - \widetilde{\sigma}(0)$ is independent of the choice of $\widetilde{\sigma}$. Let

$$d(t,t') = \widetilde{\sigma}(1) - \widetilde{\sigma}(0) - (t' - t) \in \mathbb{R}.$$

Then

$$\begin{aligned} e(d(t,t')) &= \sigma(1) - \sigma(0) - e^{2\pi i t'} + e^{2\pi i t} \\ &= s\rho(t,t')(1) - s\rho(t,t')(0) - e^{2\pi i t'} + e^{2\pi i t} \\ &= s(e^{2\pi i t}, e^{2\pi i t'})(1) - s(e^{2\pi i t}, e^{2\pi i t'})(0) - e^{2\pi i t'} + e^{2\pi i t} \\ &= e^{2\pi i t'} - e^{2\pi i t} - e^{2\pi i t'} + e^{2\pi i t} = 0. \end{aligned}$$

Therefore $d(t,t') \in \mathbb{Z}$.

To see continuity of d, first note that σ varies continuously with (t,t'). Thus $\widetilde{\sigma}(0)$ can be chosen to vary continuously with (t,t'), and hence so does $\widetilde{\sigma}(1)$, by the Homotopy Lifting Theorem.

Since $\rho(t,1) = \rho(t,0)$, the σ's associated to $(t,0)$ and $(t,1)$ are the same, and hence so are the two values of $\widetilde{\sigma}(1) - \widetilde{\sigma}(0)$. Now property (2.2) follows immediately from the change in t', and (2.3) follows similarly. Property (2.4) is clear, since both $t' - t$ and $\widetilde{\sigma}(1) - \widetilde{\sigma}(0)$ are negated when t and t' are interchanged. ∎

Since d is a continuous integer-valued function, it is constant on connected sets, a fact which we will use frequently. Note that, by (2.2) and (2.3), $(t,1)$ is in the domain of d iff $(t,0)$ is, and similarly for $(1,t)$ and $(0,t)$.

Next we provide an example of the functions d associated to three motion planning rules whose domains cover the circle. The rules for moving from z to z' are as follows.

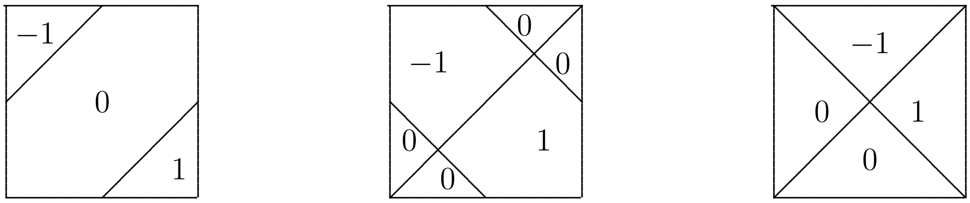
- If z and z' are not antipodal, follow the geodesic.
- If z and z' are not at the same horizontal level, let $w = \frac{z-z'}{|z-z'|}$ and $w' = -w$, and follow the geodesic from z to w, then the

path from w to w' which passes through 1, then the geodesic from w' to z'.

- If z and z' are not at the same vertical level, let $w = \frac{z-z'}{|z-z'|}$ and $w' = -w$, and follow the geodesic from z to w, then the path from w to w' which passes through $i = e^{i\pi/2}$, then the geodesic from w' to z'.

The functions d for these are as pictured in Figure 2.5:

Figure 2.5.



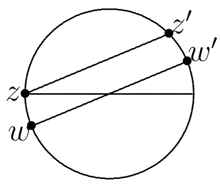
Points in ∂I^2 are in the domains except for $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ in the second and third, and $(0,\frac12)$, $(1,\frac12)$, $(\frac12,0)$, and $(\frac12,1)$ in the first and second.

For example, the region labeled "1" in the second square consists of points $(z,z') = (e^{2\pi it}, e^{2\pi it'})$ with $t > t'$ and $\frac12 < t+t' < \frac32$. One such point has $(t,t') = (\frac12,\frac18)$. For the second motion planning rule above, $w = e^{i9\pi/8}$, $w' = e^{i\pi/8}$, and σ is a counterclockwise rotation from z to z', passing through w and w'. Thus $\widetilde{\sigma}(1) - \widetilde{\sigma}(0) = \frac58$, and

$$d(\tfrac12,\tfrac18) = \tfrac58 - (\tfrac18 - \tfrac12) = 1.$$

This is illustrated in Figure 2.6.

Figure 2.6.



3. Proof of Theorem 1.1

Our proof uses the following result of general topology.

Proposition 3.1. *If W is a connected bounded open set in the plane, and K is any connected component of $\mathbb{R}^2 - W$, then its boundary, ∂K, is connected.*

This result can be found in [4, Thm 22, p. 193]. We will use the following corollary several times. It deals with a subspace U of the unit square which is open in the subspace topology. By ∂U, we mean its boundary in $\mathbb{R}^2$.

Corollary 3.2. *Let U be a connected open subset of I^2, and let P and Q be distinct points of the boundary in ∂I^2 of $U \cap \partial I^2$. Let B and B' be the two components of $\partial I^2 - \{P, Q\}$. Suppose $B \cap U = \emptyset$. Then there is a connected subset of $\partial U - (B' \cap \partial U)$ which contains P and Q.*

Proof. Apply the proposition to $W = U - (U \cap \partial I^2)$, with K being the unbounded component of $\mathbb{R}^2 - W$. Then ∂K is connected. Note that $\{P, Q\} \subset \partial K$ and $U \cap \partial I^2 \subset \partial K$. The connected component of $\partial K - (U \cap \partial I^2)$ containing P also contains Q and is contained in $\partial U - (B' \cap \partial U)$. ■

Proof of Theorem 1.1. Suppose $I \times I$ is covered by two open sets V and V' equipped with locally constant functions d and d' satisfying (2.2), (2.3), and (2.4). We will show that this leads to a contradiction, implying the theorem.

At least one of these, say V, must contain $(0, 0)$ and hence also the other three corner points. Let W_1, X_1, Y_1, and Z_1 be the connected components of V containing $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$, respectively. Note that each of W_1 and Z_1 is symmetric, while X_1 and Y_1 are symmetric to one another. It is possible that $W_1 = Z_1$; we will consider that as part of Case 2. We assume as Case 1 that ∂W_1 does not meet the edges $1 \times I$ and $I \times 1$.

Property (2.4) implies that $d(W_1) = d(Z_1) = 0$, and then properties (2.2) and (2.3) imply that $d(X_1) = -1$ and $d(Y_1) = 1$. Let

$$J_1 = \{x : (x, 0) \in W_1\} = \{y : (0, y) \in W_1\} = \{x : (x, 1) \in X_1\} = \{y : (1, y) \in Y_1\}$$

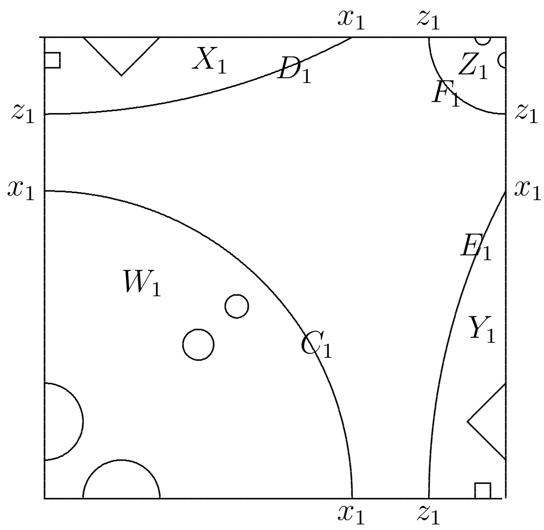
and

$$K_1 = \{x : (x, 0) \in Y_1\} = \{y : (0, y) \in X_1\} = \{x : (x, 1) \in Z_1\} = \{y : (1, y) \in Z_1\}.$$

Let $x_1 = \sup\{x : x \in J_1\}$ and $z_1 = \inf\{z : z \in K_1\}$.

By Corollary 3.2, ∂W_1 contains a connected set C_1 containing $(x_1, 0)$ and $(0, x_1)$. Similarly, there are connected sets D_1, E_1, and F_1 contained in ∂X_1, ∂Y_1, and ∂Z_1, respectively, and containing the pairs of points $(0, z_1)$ and $(x_1, 1)$, $(z_1, 0)$ and $(1, x_1)$, and $(z_1, 1)$ and $(1, z_1)$, respectively. See the illustrative schematic diagram Figure 3.3 below. The cutouts just illustrate the possibility of holes in W_1, X_1, Y_1, and Z_1.

Figure 3.3.



We claim $x_1 \le z_1$. Indeed, $d(W_1) = 0$, while $d(Y_1) = 1$, so W_1 and Y_1 are disjoint. If $x < x_1$, then C_1 is a connected set separating $(x, 0)$ from all points in Y_1. So $(x, 0)$ cannot be in Y_1. Thus $z_1 = \inf\{x : (x, 0) \in Y_1\} \ge x_1$.

If $z_1 = x_1$, then $B_1 := C_1 \cup D_1 \cup E_1 \cup F_1$ is a connected set disjoint from V. Thus V' contains an open connected set U containing B_1. Hence d' is constant on U. However $d'(x_1, 1) - d'(x_1, 0) = -1$ with both points in U, so this is a contradiction.

Thus we may assume that $x_1 < z_1$. The open set V' must contain disjoint (because of d') connected open sets W_2, X_2, Y_2, Z_2 containing C_1, D_1, E_1, and F_1, respectively. Both of the sets W_2 and Z_2 are symmetric, while X_2 and Y_2 are symmetric to one another. Let $x_2 = \sup\{x : (x, 0) \in W_2\}$ and $z_2 = \inf\{x : (x, 0) \in Y_2\}$.

By Corollary 3.2, ∂W_2 contains a connected set C_2 containing $(x_2, 0)$ and $(0, x_2)$. As before, $x_1 < x_2 \le z_2 < z_1$. Similarly, using symmetry, ∂X_2, ∂Y_2, and ∂Z_2 contain connected sets D_2, E_2, and F_2 containing the pairs of points $(0, z_2)$ and $(x_2, 1)$, $(z_2, 0)$ and $(1, x_2)$, and $(z_2, 1)$ and $(1, z_2)$, respectively.

We are in the same situation as before. If $x_2 = z_2$, then $C_2 \cup D_2 \cup E_2 \cup F_2$ is a connected set disjoint from V', and so V must contain it, yielding a contradiction on the values $d(x_2, 0)$ and $d(x_2, 1)$.

We can continue like this, obtaining connected open sets W_n, X_n, Y_n, and Z_n alternately contained in V and V', and connected sets $C_n \subset \partial W_n$ such that W_{n+1} contains C_n, and similarly for X, Y, and Z. Moreover, if $x_n = \sup\{x : (x, 0) \in W_n\}$ and $z_n = \inf\{x : (x, 0) \in Y_n\}$, then

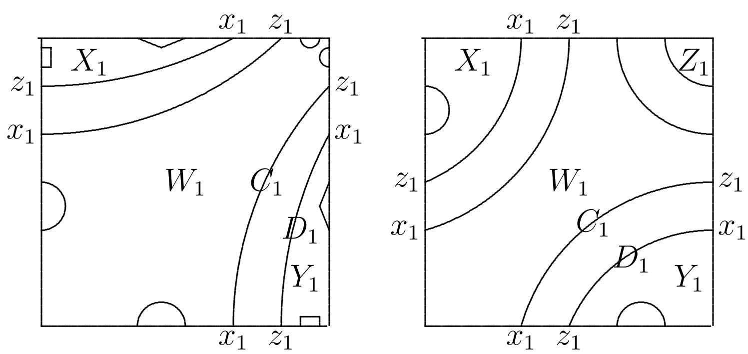
$$x_1 < x_2 < \cdots x_n \le z_n < \cdots < z_2 < z_1.$$

If, for some n, $x_n = z_n$, we stop because we have the contradiction seen above, of obtaining a connected set with contradictory d or d' values.

If it never happens that $x_n = z_n$, then we have a strictly increasing sequence of numbers x_n, which must approach a limiting value x_0. The point $(x_0, 0)$ must have a neighborhood N contained in either V or V'. WLOG, say $N \subset V$. But none of the points $(x_{2i+1}, 0)$ are in V, and yet infinitely many of them are in N. This contradiction completes the argument, showing that it is impossible to have sets V and V' as claimed.

Now we consider, as Case 2, the case in which W_1 intersects all four sides of the square. This includes the possibility that $Z_1 = W_1$. A schematic diagram of sets illustrating both possibilities for Case 2 and admitting functions d satisfying (2.2), (2.3), and (2.4) appears in Figure 3.4.

Figure 3.4.



The same argument applies to both of these possibilities. Let $x_1 = \sup\{x : (x, 0) \in W_1\}$ and $z_1 = \inf\{y : (1, y) \in W_1\}$, as in Case 1. By Corollary 3.2, ∂W_1 contains a connected set C_1 which contains $(x_1, 0)$ and $(1, z_1)$. Again $x_1 \le z_1$ since C_1 separates $(z_1, 0)$ from all $(x, 0)$ with $x < x_1$.

The second set in our cover, V', must contain a connected open set W_2 containing C_1, and also must contain its symmetric counterpart $\tau(W_2) = \{(x,y) : (y,x) \in W_2\}$. Letting d' denote the d-function for V', we have $d'(\tau(W_2)) = -d'(W_2)$ by (2.4). If $z_1 = x_1$, $d'(W_2) = d'(x_1, 0) = 1 + d'(x_1, 1) = 1 + d'(\tau(W_2))$. Hence $d'(W_2) = \frac{1}{2}$, contradicting that d' is integer-valued. Thus we may assume that $x_1 < z_1$.

Using (2.2)-(2.4), $z_1 = \inf\{x : (x,0) \in Y_1\}$ and $x_1 = \sup\{y : (1,y) \in Y_1\}$. By Corollary 3.2, ∂Y_1 contains a connected set D_1 containing $(z_1, 0)$ and $(1, x_1)$, and then V' contains a connected open set Y_2 containing D_1. Using (2.2)-(2.4), $d'(Y_2) = -d'(W_2) + 1$, so $d'(Y_2) \neq d'(W_2)$, and hence W_2 and Y_2 are disjoint.

Let $x_2 = \sup\{x : (x,0) \in W_2\}$ and $z_2 = \inf\{x : (x,0) \in Y_2\}$. Then $x_2 \le z_2$, and similarly to x_1 and z_1, we cannot have $x_2 = z_2$. We continue this process, obtaining connected sets C_n and D_n and connected open sets W_n and Y_n such that $C_n \subset \partial W_n$, $D_n \subset \partial W_n$, $C_n \subset W_{n+1}$, and $D_n \subset Y_{n+1}$, and obtain a contradiction similar to the one at the end of Case 1, using the numbers $x_n = \sup\{x : (x,0) \in W_n\}$ and $z_n = \inf\{x : (x,0) \in Y_n\}$. This completes the proof that the hypothesized sets V and V' cannot exist. ■

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