

BINOMIAL COEFFICIENTS INVOLVING INFINITE POWERS OF PRIMES

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ABSTRACT. If p is a prime and n a positive integer, let $\nu_p(n)$ denote the exponent of p in n , and $u_p(n) = n/p^{\nu_p(n)}$ the unit part of n . If α is a positive integer not divisible by p , we show that the p -adic limit of $(-1)^{p\alpha e} u_p((\alpha p^e)!)$ as $e \rightarrow \infty$ is a well-defined p -adic integer, which we call $z_{\alpha,p}$. In terms of these, we then give a formula for the p -adic limit of $\binom{ap^e+c}{bp^e+d}$ as $e \rightarrow \infty$, which we call $\binom{ap^\infty+c}{bp^\infty+d}$. Here $a \geq b$ are positive integers, and c and d are integers.

1. STATEMENT OF RESULTS

Let p be a prime number, fixed throughout. The set \mathbb{Z}_p of p -adic integers consists of expressions of the form $x = \sum_{i=0}^{\infty} c_i p^i$ with $0 \leq c_i \leq p-1$. The nonnegative integers are those x for which the sum is finite. The metric on \mathbb{Z}_p is defined by $d(x, y) = 1/p^{\nu(x-y)}$, where $\nu(x) = \min\{i : c_i \neq 0\}$. (See, e.g., [3].) The prime p will be implicit in most of our notation.

If n is a positive integer, let $u(n) = n/p^{\nu(n)}$ denote the unit factor of n (with respect to p). Our first result is

Theorem 1.1. *Let α be a positive integer which is not divisible by p . If $p^e > 4$, then*

$$u((\alpha p^{e-1})!) \equiv (-1)^{p\alpha} u((\alpha p^e)!) \pmod{p^e}.$$

Corollary 1.2. *If α is as in Theorem 1.1, then $\lim_{e \rightarrow \infty} (-1)^{p\alpha e} u((\alpha p^e)!)$ exists in \mathbb{Z}_p . We denote this limiting p -adic integer by z_α .*

If $p = 2$ or α is even, then z_α could be thought of as $u((\alpha p^\infty)!)$. It is easy for Maple to compute $z_\alpha \pmod{p^m}$ for m fairly large. For example, if $p = 2$, then $z_1 \equiv 1 + 2 + 2^3 +$

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$2^7 + 2^9 + 2^{10} + 2^{12} \pmod{2^{15}}$, and if $p = 3$, then $z_1 \equiv 1 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^4 + 3^6 + 2 \cdot 3^7 + 2 \cdot 3^8 \pmod{3^{11}}$. It would be interesting to know if there are algebraic relationships among the various z_α for a fixed prime p .

There are two well-known formulas for the power of p dividing a binomial coefficient $\binom{a}{b}$. (See, e.g., [4].) One is that

$$\nu \binom{a}{b} = \frac{1}{p-1} (d_p(b) + d_p(a-b) - d_p(a)),$$

where $d_p(n)$ denotes sum of the coefficients when n is written in p -adic form as above. Another is that $\nu \binom{a}{b}$ equals the number of carries in the base- p addition of b and $a-b$. Clearly $\nu \binom{ap^e}{bp^e} = \nu \binom{a}{b}$.

Our next result involves the unit factor of $\binom{ap^e}{bp^e}$. Here one of a or b might be divisible by p . For a positive integer n , let $z_n = z_{u(n)}$, where $z_{u(n)} \in \mathbb{Z}_p$ is as defined in Corollary 1.2.

Theorem 1.3. *Suppose $1 \leq b \leq a$, $\nu(a-b) = 0$, and $\{\nu(a), \nu(b)\} = \{0, k\}$ with $k \geq 0$. Then*

$$u \binom{ap^e}{bp^e} \equiv (-1)^{pck} \frac{z_a}{z_b z_{a-b}} \pmod{p^e},$$

$$\text{where } c = \begin{cases} a & \text{if } \nu(a) = k \\ b & \text{if } \nu(b) = k. \end{cases}$$

Since $\nu \binom{ap^e}{bp^e}$ is independent of e , we obtain the following immediate corollary.

Corollary 1.4. *In the notation and hypotheses of Theorem 1.3, in \mathbb{Z}_p*

$$\binom{ap^\infty}{bp^\infty} := \lim_{e \rightarrow \infty} \binom{ap^e}{bp^e} = p^{\nu \binom{a}{b}} (-1)^{pck} \frac{z_a}{z_b z_{a-b}}.$$

Our final result analyzes $\binom{ap^\infty + c}{bp^\infty + d}$, where c and d are integers, possibly negative.

Theorem 1.5. *If a and b are as in Theorem 1.3, and c and d are integers, then in \mathbb{Z}_p*

$$\binom{ap^\infty + c}{bp^\infty + d} := \lim_{e \rightarrow \infty} \binom{ap^e + c}{bp^e + d} = \begin{cases} \binom{ap^\infty}{bp^\infty} \binom{c}{d} & c, d \geq 0 \\ \binom{ap^\infty}{bp^\infty} \binom{c}{d} \frac{a-b}{a} & c < 0 \leq d \\ \binom{ap^\infty}{bp^\infty} \binom{c}{c-d} \frac{b}{a} & c < 0 \leq c-d \\ 0 & \text{otherwise.} \end{cases}$$

We use the standard definition that if $c \in \mathbb{Z}$ and $d \geq 0$, then

$$\binom{c}{d} = c(c-1)\cdots(c-d+1)/d!.$$

These ideas arose in work of the author extending the work in [1] and [2].

2. PROOFS

In this section, we prove the three theorems stated in Section 1. The main ingredient in the proof of Theorem 1.1 is the following lemma.

Lemma 2.1. *Let α be a positive integer which is not divisible by p . Let S denote the multiset consisting of the least nonnegative residues mod p^e of $u(i)$ for all i satisfying $\alpha p^{e-1} < i \leq \alpha p^e$. Then every positive p -adic unit less than p^e occurs exactly α times in S .*

Proof. For every p -adic unit u in $[1, \alpha p^e]$, there is a unique nonnegative integer t such that $\alpha p^{e-1} < p^t u \leq \alpha p^e$. This is true because the end of this interval equals p times its beginning. This u equals $u(p^t u)$. Thus S consists of the reductions mod p^e of all units in $[1, \alpha p^e]$. \square

Proof of Theorem 1.1. If $p^e > 4$, the product of all p -adic units less than p^e is congruent to $(-1)^p \pmod{p^e}$. (See, e.g., [4, Lemma 1], where the argument is attributed to Gauss.) The theorem follows immediately from this and Lemma 2.1, since $u((\alpha p^e)!)/u((\alpha p^{e-1}!)!$ is the product of the numbers described in Lemma 2.1. \square

Proof of Theorem 1.3. Suppose $\nu(b) = 0$ and $a = \alpha p^k$ with $k \geq 0$ and $\alpha = u(a)$. Then, mod p^e ,

$$\begin{aligned} u\left(\frac{\alpha p^{e+k}}{b p^e}\right) &= \frac{u((\alpha p^{e+k})!)}{u((b p^e)! \cdot u((a-b)p^e)!)} \\ &\equiv \frac{(-1)^{p\alpha(e+k)} z_a}{(-1)^{p b e} z_b \cdot (-1)^{p(a-b)e} z_{a-b}} \\ &= (-1)^{p a k} \frac{z_a}{z_b z_{a-b}}, \end{aligned}$$

as claimed. Here we have used 1.1 and 1.2. A similar argument works if $\nu(b) = k > 0$ (and $\nu(a) = 0$). \square

Our proof of Theorem 1.5 uses the following lemma.

Lemma 2.2. *Suppose f is a function with domain $\mathbb{Z} \times \mathbb{Z}$ which satisfies Pascal's relation*

$$(2.3) \quad f(n, k) = f(n-1, k) + f(n-1, k-1)$$

for all n and k . If $f(0, d) = A\delta_{0,d}$ for all $d \in \mathbb{Z}$ and $f(c, 0) = Ar$ for all $c < 0$, then

$$f(c, d) = \begin{cases} A\binom{c}{d} & c, d \geq 0 \\ A\binom{c}{d}r & c < 0 \leq d \\ A\binom{c}{c-d}(1-r) & c < 0 \leq c-d \\ 0 & \text{otherwise.} \end{cases}$$

The proof of this lemma is straightforward and omitted. It is closely related to work in [5] and [6], in which binomial coefficients are extended to negative arguments in a similar way. However, in that case (2.3) does not hold if $n = k = 0$.

Proof of Theorem 1.5. Fix $a \geq b > 0$. If $f_e(c, d) := \binom{ap^e+c}{bp^e+d}$, where e is large enough that $ap^e + c > 0$ and $bp^e + d > 0$, then (2.3) holds for f_e . If, as $e \rightarrow \infty$, the limit exists for two terms of this version of (2.3), then it also does for the third, and (2.3) holds for the limiting values, for all $c, d \in \mathbb{Z}$. The theorem then follows from Lemma 2.2 and (2.4) and (2.5) below, using also that if $d < 0$, then $\binom{ap^e}{bp^e+d} = \binom{ap^e}{(a-b)p^e+|d|}$, to which (2.4) can be applied.

If $d > 0$, then

$$(2.4) \quad \binom{ap^e}{bp^e+d} = \binom{ap^e}{bp^e} \frac{((a-b)p^e) \cdots ((a-b)p^e - d + 1)}{(bp^e + 1) \cdots (bp^e + d)} \rightarrow 0$$

in \mathbb{Z}_p as $e \rightarrow \infty$, since it is p^e times a factor whose p -exponent does not change as e increases through large values.

Let $c = -m$ with $m > 0$. Then

$$(2.5) \quad \binom{ap^e - m}{bp^e} = \binom{ap^e}{bp^e} \frac{((a-b)p^e) \cdots ((a-b)p^e - m + 1)}{ap^e \cdots (ap^e - m + 1)} \rightarrow \binom{ap^\infty}{bp^\infty} \frac{a-b}{a},$$

as $e \rightarrow \infty$, since

$$\frac{((a-b)p^e - 1) \cdots ((a-b)p^e - m + 1)}{(ap^e - 1) \cdots (ap^e - m + 1)} \equiv 1 \pmod{p^{e - \lceil \log_2(m) \rceil}}.$$

□

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