TOPOLOGICAL COMPLEXITY OF SOME PLANAR POLYGON SPACES

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Abstract. Using known results about their mod-2 cohomology ring, we derive strong lower bounds for the topological complexity of the space $\overline{M}_{n,r}$ of isometry classes of n-gons in the plane with one side of length r and all others of length 1, provided that $n-r$ is not an odd integer.

1. Statement of results

The topological complexity, $\mathrm{TC}(X)$, of a topological space X is, roughly, the number of rules required to specify how to move between any two points of X. A "rule" must be such that the choice of path varies continuously with the choice of endpoints. (See [2, §4].) We study $\mathrm{TC}(X)$ where $X = \overline{M}_{n,r}$ is the space of isometry classes of n-gons in the plane with one side of length r and all others of length 1. (See, e.g., [6, §9].) Thus

$$\overline{M}_{n,r} = \{(z_1, \ldots, z_n) \in (S^1)^n : z_1 + \cdots + z_{n-1} + rz_n = 0\}/O(2).$$

If we think of the sides of the polygon as linked arms of a robot, we might prefer the space $M_{n,r}$, in which we identify only under rotation, and not reflection. However, the cohomology algebra of $\overline{M}_{n,r}$ is better understood than that of $M_{n,r}$, leading to better bounds on TC.

If r is a positive real number, then $\overline{M}_{n,r}$ is an $(n-3)$-manifold unless $n-r$ is an odd integer (e.g., [6, p.314]), and hence satisfies $\mathrm{TC}(\overline{M}_{n,r}) \le 2n-5$ by [2, Cor 4.15].[1] By [5, 6.2], if $n-2k-1 < r < n-2k+1$, then $\overline{M}_{n,r}$ is diffeomorphic to $\overline{M}_{n,n-2k}$, and so

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[1] If $n-r$ is an odd integer, $\overline{M}_{n,r}$ is not a manifold but still satisfies $\mathrm{TC}(\overline{M}_{n,r}) \le 2n-5$. However, its cohomology algebra is not so well understood in this case, and so we do not study it here.

we restrict our discussion to the latter spaces. In this paper, we obtain the following strong lower bounds for $\mathrm{TC}(\overline{M}_{n,n-2k})$. Here and throughout, all congruences are mod 2, unless specifically stated to the contrary.

Theorem 1.1. *If* $\binom{n-4}{k-2}\equiv 1$, *then, for all* $n'\ge n$,

$$\mathrm{TC}(\overline{M}_{n',n'-2k})\ge\begin{cases}2n-6 & \text{if } k=2 \text{ or } n\ne 2^e+3\\ 2n-7 & \text{if } n=2^e+3.\end{cases}$$

Theorem 1.2. *If* $\binom{n-3}{k-2}\equiv 1$ *and* $\binom{n-3}{k-1}\equiv 1$, *then, for all* $n'\ge n$,

$$\mathrm{TC}(\overline{M}_{n',n'-2k})\ge 2n-6.$$

Theorem 1.3. *If* $D\ge 4$, $\binom{n-2}{k-1}\equiv 1$, $\binom{n-3}{k-2}\equiv 0$, *and* $\binom{n-D}{k-2}\equiv 1$, *then, for all* $n'\ge n$,

$$\mathrm{TC}(\overline{M}_{n',n'-2k})\ge 2n-2-D.$$

Theorem 1.4. *If* $\binom{n-3}{k-2}\equiv 1$, *then, for all* $n'\ge n$,

$$\mathrm{TC}(\overline{M}_{n',n'-2k})\ge 2n-k-4.$$

Note that these results never apply to the case $k=1$, since they require that some binomial coefficient $\binom{A}{k-2}$ be odd. The case $k=1$ is special, as $\overline{M}_{n,n-2}$ is homeomorphic to real projective space RP^{n-3}, for which the topological complexity agrees with the immersion dimension, a much-studied, but not yet fully understood, concept. See, e.g., [3], [1], or [4].

We obtain from Theorem 1.1 that $\mathrm{TC}(\overline{M}_{n,n-4})\ge 2n-6$, within 1 of the upper bound noted above. Nearly as good is

$$\mathrm{TC}(\overline{M}_{n,n-6})\ge\begin{cases}2n-7 & \text{if } n\equiv 0\ (4) \text{ or } n=2^e+3\\ 2n-6 & \text{otherwise,}\end{cases}$$

from Theorems 1.1, 1.2, and 1.4. Also, we obtain from Theorem 1.1

$$\mathrm{TC}(\overline{M}_{n,n-8})\ge\begin{cases}2n-6 & n\equiv 2,3\ (4)\\ 2n-8 & n\equiv 0,1\ (4),\end{cases}$$

except that these must be decreased by 1 if $n=2^e+3$ or 2^e+4.

We tabulate our lower bounds for $5\le k\le 16$ in a way which should generalize to larger values of k. We tabulate the number d for which we can prove $\mathrm{TC}(\overline{M}_{n,n-2k})\ge 2n-6-d$. Thus the gap between our lower bound and the upper bound noted above is $d+1$. In the following two tables, numbers denoted as 0 are from Theorem 1.1, while those denoted $0'$ are from Theorem 1.2. Doubly-primed numbers are implied by Theorem 1.3. The integers in the first column are from Theorem 1.4. Positive integers which follow the last 0 in a row are implied by the 0, using n' in Theorem 1.1. If this 0 occurs for $n = 2^e + 3$, then it and the integers following it must be increased by 1.

		$n-k$ mod 8						
	1	2	3	4	5	6	7	8
k 5	3	0	$1''$	$2''$	$0'$	0	2	4
6	4	0	0	0	0	2	4	6
7	5	0	$0'$	0	2	4	6	8
8	6	0	0	2	4	6	8	10

Table 1: d such that, away from $n = 2^e + 3$, $\mathrm{TC}(\overline{M}_{n,n-2k}) \geq 2n - 6 - d$

		$n-k$ mod 16														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
k 9	7	0	$1''$	$2''$	$3''$	$4''$	$5''$	$6''$	$0'$	0	2	4	6	8	10	12
10	8	0	0	0	0	0	0	0	0	2	4	6	8	10	12	14
11	9	0	$0'$	0	$0'$	0	$0'$	0	2	4	6	8	10	12	14	16
12	10	0	0	2	4	0	0	2	4	6	8	10	12	14	16	18
13	11	0	$1''$	$2''$	$0'$	0	2	4	6	8	10	12	14	16	18	20
14	12	0	0	0	0	2	4	6	8	10	12	14	16	18	20	22
15	13	0	$0'$	0	2	4	6	8	10	12	14	16	18	20	22	24
16	14	0	0	2	4	6	8	10	12	14	16	18	20	22	24	26

Table 2: d such that, away from $n = 2^e + 3$, $\mathrm{TC}(\overline{M}_{n,n-2k}) \geq 2n - 6 - d$

The proofs of all these results rely on the mod 2 cohomology ring $H^*(\overline{M}_{n,r}; \mathbf{Z}_2)$, first described in [6], together with the basic result that if in $H^*(X \times X)$ there is an m-fold nonzero product of classes of the form $y_i \otimes 1 + 1 \otimes y_i$, with $y_i \in H^1(X)$, then $\mathrm{TC}(X) \geq m + 1$.([2, Cor 4.40]) Throughout the paper, all cohomology groups have coefficients in $\mathbf{Z}_2$.

2. Proofs

In this section we prove Theorems 1.1, 1.2, 1.3, and 1.4. We begin by stating our interpretation of the cohomology ring $H^*(\overline{M}_{n,n-2k})$.

Theorem 2.1. *Let $k \geq 2$ and $n > 2k$.*

(1) *The algebra $H^*(\overline{M}_{n,n-2k})$ is generated by classes $R, V_1, \dots, V_{n-1}$ in $H^1(\overline{M}_{n,n-2k})$.*
(2) *The product of k distinct V_i's is 0.*
(3) *If $d \leq n-3$ and $S \subset \{1, \dots, n-1\}$ has $|S| < k$, then all monomials $R^{e_0} \prod_{i \in S} V_i^{e_i}$ with $e_i > 0$ for $i \in S$ and $\sum_{i \geq 0} e_i = d$ are equal. We denote this class by $T_{S,d}$.*
(4) *If $d \leq n-k-2$, then the set consisting of all $T_{S,d}$ with $|S| < k$ is a basis for $H^d(\overline{M}_{n,n-2k})$. Note that this includes the class $T_{\emptyset,d} = R^d$.*
(5) *If $n-k-1 \leq d \leq n-3$, then $H^d(\overline{M}_{n,n-2k})$ is spanned by all classes $T_{S,d}$ with $|S| < k$ subject to relations $\mathcal{R}_{L,d}$ for every subset L of $\{1, \dots, n-1\}$ with $n-k \leq |L| \leq d+1$. The relation $\mathcal{R}_{L,d}$ says*
$$\sum_{S \subset L} T_{S,d} = 0.$$

We often abbreviate $T_{\{i\},d}$ to $T_{i,d}$, and $T_{S,d}$ to T_S if the value of d is clear.

Proof. In [7, Theorem 1], the more general result proved in [6, Corollary 9.2] is applied to $\overline{M}_{n,n-2k}$. The first four parts of our theorem are immediate from the result stated there, although our $T_{S,d}$ notation is new. The relations stated in [7] are in the form of an ideal, whereas we prefer to make a listing of a basic set of relations. The result of [7] says that the relations in $H^*(\overline{M}_{n,n-2k})$ are the ideal generated by

$$\sum_{S \subset L} T_{S,|L|-1} \text{ for } L \subset \{1, \dots, n-1\} \text{ with } n-k \leq |L| \leq n-2. \tag{2.2}$$

Multiplying this relation by R^t gives a relation $\sum_{S \subset L} T_{S,|L|-1+t}$. This yields, in degree d, exactly all of our claimed relations. Additional relations in the ideal can be obtained by multiplying (2.2) by V_ℓ. If $\ell \notin L$, this equals our $\mathcal{R}_{S \cup \{\ell\},|L|} - \mathcal{R}_{S,|L|}$, while if $\ell \in L$, it equals 0. ■

Most of our proofs also utilize the following key result. Note that since $\overline{M}_{n,n-2k}$ is an $(n-3)$-manifold, $H^{n-3}(\overline{M}_{n,n-2k}) \approx \mathbf{Z}_2$.

Lemma 2.3. *In $H^{n-3}(\overline{M}_{n,n-2k}) \approx \mathbf{Z}_2$, any monomial in $R, V_1, \dots, V_{n-1}$ equals $\binom{n-2-t}{k-1-t}$, where t is the number of V_i's with positive exponent. Moreover, if $\binom{n-2-t}{k-1-t} \equiv 1$, $S \subset \{1, \dots, n-1\}$ with $|S| = t$, $n' \ge n$, and $d \le n-3$, then*

$$T_{S,d} \ne 0 \in H^d(\overline{M}_{n',n'-2k}).$$

Proof. The first statement was proved in [7, Theorem B]. For $d = n-3$, the second part follows from the first since the cohomology homomorphism induced by the inclusion map, $H^{n-3}(M_{n',n'-2k}) \to H^{n-3}(M_{n,n-2k})$, sends $T_{S,n-3}$ to $T_{S,n-3} \ne 0$. If $d < n-3$, then $T_{S,d}$ is a divisor of the nonzero class $T_{S,n-3}$, and hence is nonzero. ∎

Proof of first case of Theorem 1.1. The component of

$$(V_1 \otimes 1 + 1 \otimes V_1)^{n-3}(V_2 \otimes 1 + 1 \otimes V_2)^{n-4}$$

in $H^{n-3}(\overline{M}_{n,n-2k}) \otimes H^{n-4}(\overline{M}_{n,n-2k})$ is

(2.4)

$$V_1^{n-3} \otimes V_2^{n-4} + (n-3)V_1V_2^{n-4} \otimes V_1^{n-4} + \sum_{i=2}^{n-4} \binom{n-3}{i}\binom{n-4}{n-3-i} V_1^i V_2^{n-3-i} \otimes V_1^{n-3-i} V_2^{i-1}.$$

Note that, for $2 \le i \le n-4$, $V_1^i V_2^{n-3-i} \otimes V_1^{n-3-i} V_2^{i-1} = T_{\{1,2\},n-3} \otimes T_{\{1,2\},n-4}$, and

$$(2.5) \quad \sum_{i=2}^{n-4} \binom{n-3}{i}\binom{n-4}{n-3-i} \equiv 1 + (n-3) + \binom{2n-7}{n-3} \equiv \begin{cases} 0 & \text{if } n = 2^e + 3 \\ n & \text{otherwise.} \end{cases}$$

If $n = 2^e + 3$, (2.4) equals $V_1^{n-3} \otimes V_2^{n-4}$. By Lemma 2.3, $V_1^{n-3} = \binom{2^e}{k-2}$, which is 0 unless $k = 2$, in which case V_2^{n-4} is also nonzero.

For $n \ne 2^e + 3$, using 2.1 and (2.5), (2.4) equals

(2.6)

$$T_{1,n-3} \otimes T_{2,n-4} + nT_{\{1,2\},n-3} \otimes (T_{1,n-4} + T_{\{1,2\},n-4}) + T_{\{1,2\},n-3} \otimes T_{1,n-4}.$$

If $\binom{n-3}{k-2} \equiv 1$, then $\binom{n-4}{k-3} \equiv 0$, so $T_{1,n-3}$ and hence also $T_{2,n-4}$ are nonzero, while $T_{\{1,2\},n-3} = 0$, all by Lemma 2.3, and so (2.6) is nonzero.

If, on the other hand, $\binom{n-3}{k-2} \equiv 0$, then $\binom{n-4}{k-3} \equiv 1$ by Pascal's formula, so $T_{1,n-3} = 0$ and the first factor of the other terms is nonzero by Lemma 2.3. Since $\binom{n-4}{k-2} \not\equiv 0$, the second part of 2.3 implies that $T_{1,n-4} \ne 0$. Since $T_{\{1,2\},n-3} \ne 0$ in $H^*(\overline{M}_{n,n-2k})$, we must also have $T_{\{1,2\},n-4} \ne 0$. If n is even, (2.6) is the nonzero term $g \otimes T_1$, while if n is odd, it is the nonzero term $g \otimes T_{\{1,2\}}$. Here g is the nonzero element of $H^{n-3}(\overline{M}_{n,n-2k})$.

For $n' > n$, the cohomology homomorphism induced by the inclusion $\overline{M}_{n,n-2k} \to \overline{M}_{n',n'-2k}$ implies the result for n'. ■

Proof of Theorem 1.2. We may assume that n is even, since if n is odd, it is impossible to have both $\binom{n-3}{k-2} \equiv 1$ and $\binom{n-3}{k-1} \equiv 1$. Similarly to the proof of Theorem 1.1, we analyze the $H^{n-3} \otimes H^{n-4}$-component of

$$(R \otimes 1 + 1 \otimes R)^{n-3}(V_1 \otimes 1 + 1 \otimes V_1)^{n-4}.$$

Similarly to that proof, since $n \neq 2^e + 3$, this equals

$$R^{n-3} \otimes T_{1,n-4} + T_{1,n-3} \otimes R^{n-4}.$$

(The second term has coefficient $n - 3 \equiv 1$.) Using Lemma 2.3, the hypotheses imply that $T_{1,n-3}$ and R^{n-4} are nonzero, and also that $\binom{n-2}{k-1} \equiv 0$ and hence $R^{n-3} \equiv 0$. ■

Proof of Theorem 1.3. We prove that the $H^{n-3} \otimes H^{n-D}$-component of

$$(R \otimes 1 + 1 \otimes R)^{n-3}(V_1 \otimes 1 + 1 \otimes V_1)^{n-D}$$

is nonzero. This component equals

$$R^{n-3} \otimes T_{1,n-D} + \binom{n-3}{D-3} T_{1,n-3} \otimes R^{n-D} + \varepsilon T_{1,n-3} \otimes T_{1,n-D},$$

with $\varepsilon \in \mathbf{Z}_2$. The hypotheses imply that $R^{n-3} \neq 0$, $T_{1,n-3} = 0$, and $T_{1,n-D} \neq 0$. ■

Proof of Theorem 1.4. The component in $H^{n-3} \otimes H^{n-k-2}$ of

$$(V_1 \otimes 1 + 1 \otimes V_1)^{n-3}(V_2 \otimes 1 + 1 \otimes V_2)^{n-k-2} \tag{2.7}$$

is

$$T_{1,n-3} \otimes T_{2,n-k-2} + \varepsilon_1 T_{\{1,2\},n-3} \otimes T_{1,n-k-2} + \varepsilon_2 T_{\{1,2\},n-3} \otimes T_{\{1,2\},n-k-2},$$

with $\varepsilon_i \in \mathbf{Z}_2$. Since $T_{1,n-3} \neq 0$ by Lemma 2.3 and the hypothesis, and all monomials $T_{S,n-k-2}$ are linearly independent by Theorem 2.1(4), we deduce that our class is nonzero. ■

Proof of second case of Theorem 1.1. We show that the $H^{n-4}\otimes H^{n-4}$-component of

$$(V_1\otimes 1+1\otimes V_1)^{n-4}(V_2\otimes 1+1\otimes V_2)^{n-4}$$

is nonzero whenever $\binom{n-4}{k-2}\equiv 1$. It equals

$$V_1^{n-4}\otimes V_2^{n-4}+V_2^{n-4}\otimes V_1^{n-4}+\sum_{i=1}^{n-5}\binom{n-4}{i}^2 V_1^iV_2^{n-4-i}\otimes V_1^{n-4-i}V_2^i.$$

The summation part is $(\sum_{i=1}^{n-5}\binom{n-4}{i})T_{\{1,2\}}\otimes T_{\{1,2\}}=0$. Thus our term is nonzero if T_1 and T_2 are linearly independent in $H^{n-4}(\overline{M}_{n,n-2k})$, and this follows from Lemma 2.8 below. ■

Lemma 2.8. *If $\binom{n-4}{k-2}\equiv 1$, then there is a homomorphism $H^{n-4}(\overline{M}_{n,n-2k})\to \mathbf{Z}_2$ sending T_1 to 1, other T_i to 0, and R^{n-4} to 0.*

Proof. For reasons of simplicity and symmetry, we seek a homomorphism ϕ that sends T_S to 0 if $1\notin S$, while if $1\in S$, then $\phi(T_S)=x_{|S|}$ with $x_1=1$, and other x_r, $2\le r\le k-1$, are elements of $\mathbf{Z}_2$ to be determined. By Theorem 2.1(5), these x_r's must satisfy that

$$\sum_{r=1}^{k-1}x_r\binom{|L|-1}{r-1}=0 \text{ for } n-k\le |L|\le n-3.$$

The binomial coefficients here give the number of r-subsets of L which contain 1, assuming $1\in L$.

Let $x_r=\binom{n-3-r}{k-1-r}$. Since $\binom{n-4}{k-2}\equiv 1$, the desired condition becomes

$$\sum_{r=1}^{k-1}\binom{n-3-r}{k-1-r}\binom{\ell}{r-1}=0 \text{ for } n-k-1\le \ell\le n-4.$$

Using $\binom{-a}{b}=\pm\binom{a+b-1}{b}$, this condition becomes $\sum\binom{-(n-1-k)}{k-1-r}\binom{\ell}{r-1}=0$, where the sum is taken over all values of r. The LHS equals $\binom{\ell-n+1+k}{k-2}$, with the range of values of the top part of this binomial coefficient ranging from 0 to $k-3$, inclusive, verifying the claim. ■

Other investigations similar to those of these proofs have not yielded any additional TC results.

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