BOUNDS FOR HIGHER TOPOLOGICAL COMPLEXITY OF REAL PROJECTIVE SPACE IMPLIED BY BP

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Abstract. We use Brown-Peterson cohomology to obtain lower bounds for the higher topological complexity, $\mathrm{TC}_k(RP^{2m})$, of real projective spaces, which are often much stronger than those implied by ordinary mod-2 cohomology.

1. Introduction and main results

In [8], Farber introduced the notion of topological complexity, $\mathrm{TC}(X)$, of a topological space X. This can be interpreted as one less than the minimal number of rules, called *motion planning rules*, required to tell how to move between any two points of X.[1] This became central in the field of topological robotics when X is the space of configurations of a robot or system of robots. This was generalized to higher topological complexity, $\mathrm{TC}_k(X)$, by Rudyak in [10]. This can be thought of as one less than the number of rules required to tell how to move consecutively between any k specified points of X ([10, Remark 3.2.7]). In [2], the study of $\mathrm{TC}_k(P^n)$ was initiated, and this was continued in [6], where the best lower bounds implied by mod-2 cohomology were obtained. Here P^n denotes real projective space.

Since $\mathrm{TC}_2(P^n)$ is usually equal to the immersion dimension ([9]), and a sweeping family of strong nonimmersion results was obtained using Brown-Peterson cohomology, $BP^*(-)$, in [3], one is led to apply BP to obtain lower bounds for $\mathrm{TC}_k(P^n)$ for $k > 2$. In this paper, we obtain a general result, Theorem 1.1, which implies lower

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[1] Farber's original definition did not include the "one less than" part, but most recent papers have defined it as we have done here.

bounds in many cases, and then focus in Theorem 1.4 on a particular family of cases, which we show is often much stronger than the results implied by mod-2 cohomology.

The general result is obtained from known information about the BP-cohomology algebra of products of real projective spaces. It gives conditions under which nonzero classes of a certain form can be found. Here and throughout, $\nu(-)$ denotes the exponent of 2 in an integer.

Theorem 1.1. *Let $k \geqslant 3$ and $r \geqslant 0$. Suppose there are positive integers $a_1, \ldots, a_{k-1}$ whose sum is $km - (2^k - 1)2^r$ such that*

$$\nu\left(\prod_{i=1}^{k-1} \tbinom{a_i}{j_i}\right) \geqslant 2^r \tag{1.2}$$

for all $j_1, \ldots, j_{k-1}$ with $j_i \leqslant m$ and $\sum_{i=1}^{k-1} j_i \geqslant (k-1)m - (2^k - 1)2^r$. Suppose also that

$$\nu\left(\sum_{\ell} \prod_{i=1}^{k-1} \tbinom{a_i}{m-\ell_i}\right) = 2^r, \tag{1.3}$$

where $\ell = (\ell_1, \ldots, \ell_{k-1})$ ranges over all $(k-1)$-tuples of the k distinct numbers 2^{r+t}, $0 \leqslant t \leqslant k-1$. Then

$$\mathrm{TC}_k(P^{2m}) \geqslant 2km - (2^k - 1)2^{r+1}.$$

Theorem 1.1 applies in many cases, but we shall focus on one family. Here and throughout, $\alpha(-)$ denotes the number of 1's in the binary expansion of an integer.

Theorem 1.4. *Suppose $k \geqslant 3$, $r \geqslant k-3$, and $m = A \cdot 2^r$ with $A \geqslant 2^{k-1}$. Then*

$$\mathrm{TC}_k(P^{2m}) \geqslant 2km - (2^k - 1)2^{r+1}$$

if

- a. *$k = 3$ and either*
  - i. *$A \equiv 5\ (8)$ and $\alpha(A) = 2^r + 2$, or*
  - ii. *$A \equiv 2\ (4)$ and $\alpha(A) = 2^r + 2$; or*
- b. *$k \geqslant 4$ and either*
  - i. *$A \equiv 6\ (8)$ and $\alpha(A) = 2^r + 2$, or*
  - ii. *$A \equiv 3\ (8)$ and $\alpha(A) = 2^r + 3$.*

We prove Theorems 1.1 and 1.4 in Section 2. In Section 3, we describe more specifically some families of particular values of (m, k, r) to which this result applies, and the extent to which these results are much stronger than those implied by mod-2 cohomology. In Section 4, we prove that the cohomology-implied bounds for $\mathrm{TC}_k(P^n)$ are constant for long intervals of values of n. In these intervals, the BP-implied bounds become much stronger than those implied by cohomology.

2. Proofs of main theorems

In this section, we prove Theorems 1.1 and 1.4. The first step, Theorem 2.1, follows suggestions of Jesus González, and is similar to work in [2]. We are very grateful to González for these suggestions. There are canonical elements $X_1, \ldots, X_k$ in $BP^2((P^n)^k)$, where $(P^n)^k$ is the Cartesian product of k copies of P^n.

Theorem 2.1. *If* $(X_1 - X_k)^{a_1} \cdots (X_{k-1} - X_k)^{a_{k-1}} \neq 0 \in BP^*((P^n)^k)$, *then*

$$TC_k(P^n) \geqslant 2a_1 + \cdots + 2a_{k-1}.$$

Proof. Let $(P^n)^{[0,1]}$ denote the space of paths in P^n, and

$$P_{n,k} = (S^n)^k/((z_1, \ldots, z_k) \sim (-z_1, \ldots, -z_k))$$

a projective product space.([5]) The quotient map $P_{n,k} \xrightarrow{\pi} (P^n)^k$ is a $(\mathbb{Z}_2)^{k-1}$-cover, classified by a map $(P^n)^k \xrightarrow{\mu} B((\mathbb{Z}_2)^{k-1}) = (P^\infty)^{k-1}$. The map $(P^n)^{[0,1]} \xrightarrow{p} (P^n)^k$ defined by

$$\sigma \mapsto (\sigma(0), \sigma(\tfrac{1}{k-1}), \ldots, \sigma(\tfrac{k-2}{k-1}), \sigma(1))$$

lifts to a map $(P^n)^{[0,1]} \xrightarrow{\widetilde{p}} P_{n,k}$.([2, (3.2)]) A definition of $\mathrm{TC}_k(P^n)$ is as the sectional category $\mathrm{secat}(p)$. The lifting $\widetilde{p}$ implies that $\mathrm{secat}(p) \geqslant \mathrm{secat}(\pi)$.

Let $G = (\mathbb{Z}_2)^{k-1}$, and $B_tG = (\ast^{t+1}G)/G$, where $\ast^{t+1}G$ denotes the iterated join of $t + 1$ copies of G. Note that B_tG is the tth stage in Milnor's construction of BG, with a map $i_t : B_tG \to BG$. By [11, Thm 9, p. 86], as described in [2, (4.1)], μ lifts to a map $(P^n)^k \xrightarrow{\widetilde{\mu}} B_{\mathrm{secat}(\pi)}G$.

$$\begin{tikzcd}
(P^n)^{[0,1]} \arrow[r, "\widetilde{p}"] \arrow[dr, "p"'] & P_{n,k} \arrow[d, "\pi"] & B_{\mathrm{secat}(\pi)}G \arrow[d, "i_{\mathrm{secat}(\pi)}"] \\
& (P^n)^k \arrow[r, "\mu"] \arrow[ur, "\widetilde{\mu}"] & BG = (P^\infty)^{k-1}
\end{tikzcd}$$

By [2, Prop 3.1], μ classifies $(p_1^*(\xi)\otimes p_k^*(\xi))\times\cdots\times(p_{k-1}^*(\xi)\otimes p_k^*(\xi))$, and so, by [1, Prop 3.6], the induced homomorphism

$$BP^*((P^\infty)^{k-1}) \xrightarrow{\mu^*} BP^*((P^n)^k)$$

satisfies $\mu^*(X_i)=u_i(X_i-X_k)$ for $1\leqslant i\leqslant k-1$, with u_i a unit. Since $\mu^*=\widetilde{\mu}^* i^*_{\mathrm{secat}(\pi)}$ and B_tG is t-dimensional, $\mu^*(X_1^{a_1}\cdots X_{k-1}^{a_{k-1}})=0$ if $2a_1+\cdots+2a_{k-1}>\mathrm{secat}(\pi)$. The theorem now follows since $\prod(X_i-X_k)^{a_i}\neq 0$ implies $\mu^*(\prod X_i^{a_i})\neq 0$, which implies

$$\sum 2a_i\leqslant \mathrm{secat}(\pi)\leqslant \mathrm{secat}(p)=\mathrm{TC}_k(P^n).$$

■

We use this to prove Theorem 1.1.

Proof of Theorem 1.1. Let I denote the ideal $(v_0,\ldots,v_k)\subset BP^*$. Recall $v_0=2$ and $|v_i|=2(2^i-1)$. In $BP^*(X)$, let F_s denote the BP^*-submodule $I^s\cdot BP^*(X)$. It follows from [12, 2.2], [4, Cor 2.4], and [7, Thm 1.10] that in $BP^*((P^{2m})^k)$, for $r\geqslant 0$ and integers $j_1,\ldots,j_k$,

$$2^{2^r}X_1^{j_1}\cdots X_k^{j_k}\equiv v_k^{2^r}\sum X_1^{j_1+\ell_1}\cdots X_k^{j_k+\ell_k}\quad \mathrm{mod}\ F_{2^r+1},\tag{2.2}$$

where the sum is taken over all permutations $(\ell_1,\ldots,\ell_k)$ of $\{2^r,\ldots,2^{r+k-1}\}$. (An analogous result was derived in BP-homology in [7], following similar, but not quite so complete, results in [12] and [4], which also discussed the dualization to obtain BP-cohomology results.)

The result follows from Theorem 2.1 once we show that

$$(X_1-X_k)^{a_1}\cdots(X_{k-1}-X_k)^{a_{k-1}}\neq 0\in BP^{2km-(2^k-1)2^{r+1}}((P^{2m})^k).$$

This expands as $\displaystyle\sum_{j_1,\ldots,j_{k-1}}\pm\binom{a_1}{j_1}\cdots\binom{a_{k-1}}{j_{k-1}}X_1^{j_1}\cdots X_{k-1}^{j_{k-1}}X_k^{km-(2^k-1)2^r-j_1-\cdots-j_{k-1}}$, for values of $j_1,\ldots,j_{k-1}$ described in Theorem 1.1. By (2.2) and (1.2), this equals, mod F_{2^r+1},

$$v_k^{2^r}\sum_{j_1,\ldots,j_{k-1}}\sum_{\ell}\pm 2^{-2^r}\binom{a_1}{j_1}\ldots\binom{a_{k-1}}{j_{k-1}}X_1^{j_1+\ell_1}\cdots X_{k-1}^{j_{k-1}+\ell_{k-1}}X_k^{km-j_1-\ell_1-\cdots-j_{k-1}-\ell_{k-1}},\tag{2.3}$$

with $\ell=(\ell_1,\ldots,\ell_{k-1})$ as in (1.3). Note here that $\ell_k=2^{r+k}-2^r-\ell_1-\cdots-\ell_{k-1}$. The terms in (2.3) are 0 unless the exponent of each X_i equals m, since otherwise there would be a factor X^p with $p > m$. We are left with

$$\Big(\sum_\ell \pm 2^{-2^r}\tbinom{a_1}{m-\ell_1}\cdots\tbinom{a_{k-1}}{m-\ell_{k-1}}\Big)v_k^{2^r}X_1^m\cdots X_k^m$$

with $(\ell_1,\dots,\ell_{k-1})$ as above, and this is nonzero by the hypothesis (1.3) and the fact, as was noted in [12], that by the (proven) Conner-Floyd conjecture, $v_k^h X_1^m\cdots X_k^m \neq 0$ for any nonnegative integer h. ■

In the following proof of Theorem 1.4, we will often use without comment Lucas's Theorem regarding binomial coefficients mod 2, and that

$$\nu\tbinom{m}{n} = \alpha(n)+\alpha(m-n)-\alpha(m), \text{ and } \alpha(x-1) = \alpha(x)-1+\nu(x). \quad (2.4)$$

Proof of Theorem 1.4. We explain the proof when $k \geqslant 4$ and $A \equiv 6$ (8), and then describe the minor changes required when $A \equiv 3$ or $k = 3$. We apply Theorem 1.1 with

$$a_i = m-(2^k-1)2^{r-i},\ 1 \leqslant i \leqslant k-3,\ a_{k-2} = m, \text{ and } a_{k-1} = 2m-(2^k-1)2^{r-(k-3)}.$$

For (1.2), we show

$$\nu\tbinom{a_{k-1}}{j} \geqslant 2^r \text{ if } (k-1)m-(2^k-1)2^r-(a_1+\cdots+a_{k-2}) \leqslant j \leqslant m.$$

Thus we are considering $\nu\binom{2m-(2^k-1)2^{r-(k-3)}}{j}$ with $m-(2^k-1)2^{r-(k-3)} \leqslant j \leqslant m$. By symmetry, we may restrict to $m-(2^{k-1}-1)2^{r-(k-3)} \leqslant j \leqslant m$. Let $m = (8B+6)2^r$ with $\alpha(B) = 2^r$. We first restrict to j's divisible by $2^{r-(k-3)}$; let $j = 2^{r-(k-3)}h$. Now we are considering $\nu\binom{(8B+6)2^{k-2}-2^k+1}{h}$ with $2^{k-3}(8B+6)-(2^{k-1}-1) \leqslant h \leqslant 2^{k-3}(8B+6)$. Lemma 2.5 with $t = k-2$ shows that $\nu\binom{(8B+6)2^{k-2}-2^k+1}{h} \geqslant \alpha(B)$ for the required values of h. The proof for arbitrary j (in the required range) follows from the easily proved fact that

$$\text{for } 0 < \delta < 2^k, \quad \nu\tbinom{N\cdot 2^k}{M\cdot 2^k+\delta} > \nu\tbinom{N\cdot 2^k}{M\cdot 2^k}.$$

Now we prove (1.3). We divide the top and bottom of the binomial coefficients by $2^{r-(k-3)}$; this does not change the exponent. The tops are now

$$2^{k-3}A-(2^k-1)2^{k-4},\dots,2^{k-3}A-(2^k-1)2^0,\ 2^{k-3}A,\ 2^{k-2}A-(2^k-1),$$

and the bottoms are selected from $2^{k-3}A-2^{k-3},\dots,2^{k-3}A-2^{2k-4}$. All the bottoms except the last one are greater than the first top one. Thus to get a nonzero product in (1.3), the last bottom must accompany the first top, and after dividing top and bottom by 2^{k-4}, it becomes $\binom{2A-(2^k-1)}{2A-2^k} \equiv 1 \bmod 2$. Similar considerations work inductively for all but the final two factors, showing that the ith bottom from the end must appear beneath the ith top and gives an odd factor. What remains is

$$\sum \binom{2^{k-3}A}{j}\binom{2^{k-2}A-2^k+1}{j'},$$

where (j,j') are the ordered pairs of distinct elements of

$$\{2^{k-3}A-2^{k-3}, 2^{k-3}A-2^{k-2}, 2^{k-3}A-2^{k-1}\}.$$

The $+1$ on top does not affect the exponent of the binomial coefficients, and so we may remove it and then divide tops and bottoms by 2^{k-3}, obtaining $\sum \binom{A}{j}\binom{2A-8}{j'}$, where (j,j') are ordered pairs of $A-1$, $A-2$, and $A-4$.

If $A \equiv 6 \bmod 8$, $\nu\binom{A}{j} = 0$ if $j = A-2$ or $A-4$, and is > 0 if $j = A-1$. Also, with $A = 8B+6$, $\nu\binom{2A-8}{j'} = \alpha(B)$ if $j' = A-2$, and is $> \alpha(B)$ if $j' = A-1$ or $A-4$. Thus the sum in (1.3) has $\nu(-) = 2^r$, coming from the single summand corresponding to $(j,j') = (A-4, A-2)$.

When $A \equiv 3 \bmod 8$, the following minor changes must be made in the above argument. Let $A = 8B+3$. A minimal value of $\nu\binom{a_{k-1}}{j}$ occurs when $j = 2^{r-(k-3)}h$ with $h = 2^{k-3}(8B+3)-2^{k-3}$. We obtain $\nu\binom{16B-2}{8B+2} = \alpha(B)-1 = 2^r$ since $\alpha(A) = 2^r+3$. For (1.3), the minimal value $\nu\left(\binom{A}{j}\binom{2A-8}{j'}\right) = 2^r$ occurs only for $(j,j') = (A-2, A-1)$.

Part (a) of Theorem 1.4 follows similarly. We have $a_1 = m$ and $a_2 = 2m - 7\cdot 2^r$. Then by the same methods as used above, we show that with m as in the theorem, and P denoting a positive number and I a number which is irrelevant,

- If $m - 7\cdot 2^r \leqslant j \leqslant m$, then $\nu\binom{2m-7\cdot 2^r}{j} \geqslant 2^r$.
- The values $(\nu\binom{m}{m-2^r}, \nu\binom{m}{m-2^{r+1}}, \nu\binom{m}{m-2^{r+2}})$ are $(0,P,0)$ (resp. $(P,0,I)$) in case (i) (resp. (ii)) of the theorem.
- The values $(\nu\binom{2m-7\cdot 2^r}{m-2^r} - 2^r, \nu\binom{2m-7\cdot 2^r}{m-2^{r+1}} - 2^r, \nu\binom{2m-7\cdot 2^r}{m-2^{r+2}}) - 2^r)$ are $(P,0,0)$ (resp. $(0,0,P)$) in case (i) (resp. (ii)) of the theorem.

■

The following lemma was used above.

Lemma 2.5. *If $t \geqslant 2$ and $-2^t+1 \leqslant d \leqslant 2^t$, then* $\nu\binom{(8B+2)2^t+1}{(4B+2)2^t+d} \geqslant \alpha(B)$.

Proof. Using (2.4), we can show

$$\nu\binom{(8B+2)2^t+1}{(4B+2)2^t+d}=\begin{cases}\alpha(B)+t+1-\nu(d(d-1)) & -2^t+1\leqslant d<0\\ \alpha(B) & d=0,1\\ \alpha(B)+t+\nu(B)+2-\nu(d(d-1)) & 2\leqslant d\leqslant 2^t,\end{cases}$$

from which the lemma is immediate. ∎

3. Numerical results

In this section, we compare the lower bounds for $\mathrm{TC}_k(P^{2m})$ implied by BP with those implied by mod-2 cohomology. In [6], the best lower bounds obtainable using mod-2 cohomology were obtained. They are restated here in (4.2). In Table 1, we compare these with the results implied by our Theorems 1.1 and 1.4 for $\mathrm{TC}_3(P^{2m})$ with $32\leqslant m<63$. Results in the BP column are those implied by 1.1, and those indicated with an asterisk are implied by 1.4. It is quite possible that there are additional results implied by Theorem 2.1, since Theorem 1.1 takes into account only one type of implication about nonzero classes in $BP^*((P^n)^k)$. Note that the BP-bounds are significantly stronger in the second half of the table.

TABLE 1. Lower bounds for $\mathrm{TC}_3(P^{2m})$ implied by $H^*(-)$ and by BP

m	$H^*(-)$	BP
32	192	152
33	198	152
34	204	190
35	206	190
36	216	190
37	222	208*
38	222	214*
39	222	214*
40	240	214*
41	246	232
42	252	238*
43	254	238*
44	254	238*
45	254	238*
46	254	248
47	254	248
48	254	248
49	254	280
50	254	286*
51	254	286*
52	254	286*
53	254	304
54	254	310
55	254	310
56	254	310
57	254	310
58	254	320*
59	254	320*
60	254	332*
61	254	332*
62	254	332*
63	254	332*

In Table 2, we present another comparison of the results implied by Theorem 1.4 and those implied by ordinary mod-2 cohomology. We consider lower bounds for $\mathrm{TC}_4(P^{2m})$ for $2^{11} \leqslant m < 2^{12}$. In Table 2, the first column refers to a range of values of m, the second column to the number of distinct new results implied by Theorem 1.4 in that range, and the third column to the range of the ratio of bounds implied by Theorem 1.4 to those implied by ordinary cohomology. There are many other stronger bounds implied by BP via Theorem 1.1, but our focus here is on the one family which we have analyzed for all k and r.

TABLE 2. Ratio of lower bounds for $\mathrm{TC}_4(P^{2m})$ implied by Theorem 1.4 to those implied by $H^*(-)$

m	#	ratio
$[2048, 2815]$	29	$[.9620, 1.0384]$
$[2816, 3071]$	7	$[.9877, 1.0673]$
$[3072, 3979]$	26	$[.9783, 1.2700]$
$[3980, 4095]$	1	1.2908

In the range $2816 \leqslant m \leqslant 3071$ here, the bound for $\mathrm{TC}_4(P^{2m})$ implied by mod-2 cohomology is constant at 22525, while that implied by Theorem 1.4 increases from 22248 to 24040. In the longer range $3072 \leqslant m \leqslant 4095$ here, the bound for $\mathrm{TC}_4(P^{2m})$ implied by mod-2 cohomology is constant at 24573, while that implied by Theorem 1.4 increases from 24040 to 31720. Next, we examine what happens in the generalization of this latter range to $\mathrm{TC}_k(P^{2m})$ for arbitrary k and arbitrary 2-power near the end of the range. In Theorem 4.1, we will show that the bound for $\mathrm{TC}_k(P^{2m})$ implied by cohomology has the constant value $(k-1)(2^e-1)$ for $[\frac{k-1}{k}\cdot 2^e] \leqslant 2m \leqslant 2^e-1$.

In this range, the bound implied by Theorem 1.4 will increase from a value approximately equal to the cohomology-implied bound to a value which, as we shall explain, is asymptotically as much greater than the cohomology-implied bound as it could possibly be. The following result gives a result at the end of each 2-power interval, since each e can be written uniquely as $2^r + r + 3 + d$ for $0 \leqslant d \leqslant 2^r$. For example, the case $r = 1$, $d = 0$, $k = 3$ in this proposition is the 332$*$ next to $m = 60$ in Table 1, and the case $r = 2$, $d = 3$, $k = 4$ gives $m = 3980$, the start of the last row of Table 2.

Proposition 3.1. *For $r \geqslant 1$ and $0 \leqslant d \leqslant 2^r$, let*

$$m = \begin{cases} 2^{r+1}(2^{2^r+2}-1) & d = 0,\ k \geqslant 3 \\ 2^{r+d+2}(2^{2^r+1}-1) + 2^{r+1} & d > 0,\ k = 3 \\ 2^{r+d+2}(2^{2^r+1}-1) + 3\cdot 2^r & d > 0,\ k > 3. \end{cases}$$

Then $\mathrm{TC}_k(P^{2m}) \geqslant 2km - (2^k - 1)2^{r+1}$.

Proof. It is straightforward to check that the conditions of Theorem 1.4 are satisfied for these values of m and r. ■

For m as in Proposition 3.1, the lower bound for $\mathrm{TC}_k(P^{2m})$ implied by cohomology is $(k-1)(2^{2^r+r+4+d} - 1)$. One can check that the ratio of the bound in Proposition 3.1 to the cohomology bound is greater than

$$\frac{k}{k-1} - \frac{1}{2^{2^r+1}}.$$

Since, as was noted in [2], $(k-1)n \leqslant \mathrm{TC}_k(P^n) \leqslant kn$, the largest the ratio of any two estimates of $\mathrm{TC}_k(P^n)$ could possibly be is $k/(k-1)$. Thus the BP-bound improves on the cohomology bound asymptotically by as much as it possibly could, as e (hence r) becomes large.

Jesus González ([2]) has particular interest in estimates for $\mathrm{TC}_k(P^{3\cdot 2^e})$. We shall prove the interesting fact that our Theorems 1.1 and 1.4 improve significantly on the cohomological lower bound for $\mathrm{TC}_3(P^{3\cdot 2^e})$, but not for $\mathrm{TC}_k(P^{3\cdot 2^e})$ when $k > 3$.

The bound implied by cohomology (Theorem 4.1) is

$$\mathrm{TC}_k(P^{3\cdot 2^e}) \geqslant (k-1)(2^{e+2} - 1). \tag{3.2}$$

Since $2km - (2^k - 1)2^{r+1} \leqslant (k-1)(2^{e+2} - 1)$ if $k \geqslant 4$ and $m \leqslant 3 \cdot 2^e$ (and $r \geqslant 0$), Theorem 1.1 cannot possibly improve on (3.2) if $k \geqslant 4$. In order for BP to possibly improve on (3.2) when $k \geqslant 4$, a much more delicate analysis of $BP^*((P^n)^k)$ would have to be performed, involving new ways of showing that classes are nonzero, and then using Theorem 2.1.

However, Theorem 1.4 implies a lower bound for $\mathrm{TC}_3(P^{3\cdot 2^e})$ which is asymptotically 9/8 times the bound in (3.2).

Theorem 3.3. *Let* $r \geqslant 1$, $0 \leqslant d \leqslant 2^r$, *and* $e = 2^r + r + d + 3$. *Then*

$$\mathrm{TC}_3(P^{3\cdot 2^e}) \geqslant 9 \cdot 2^e - 3 \cdot 2^{r+3+d} - 2^{r+1}.$$

Proof. One easily checks that, with e as in the theorem, $m = 3 \cdot 2^{e-1} - 2^{r+2+d} + 2^{r+1}$ satisfies the hypothesis of Theorem 1.4(a)(ii), and that Theorem 1.4 then implies $\mathrm{TC}_3(P^{2m}) \geqslant 9 \cdot 2^e - 3 \cdot 2^{r+3+d} - 2^{r+1}$, implying this theorem by naturality. ■

In Table 3, we compare the bounds for $\mathrm{TC}_3(P^{3\cdot 2^e})$ implied by Theorem 3.3 and by (3.2) for various values of e. Every e has a unique r and d. The m-column is the value of $m < 3\cdot 2^{e-1}$ which appears in the proof of 3.3. The "BP-bound" column is the bound for $\mathrm{TC}_3(P^{3\cdot 2^e})$ given by Theorem 3.3, and the "H^*-bound" column that is given by (3.2). The final column is the ratio of the BP-bound to the H^*-bound, which approaches 1.125 as e gets large.

TABLE 3. Ratio of lower bounds for $\mathrm{TC}_3(P^{3\cdot 2^e})$ implied by Theorem 3.3 to those implied by $H^*(-)$

e	r	d	m	BP-bound	H^*-bound	ratio
6	1	0	92	524	510	1.027
7	1	1	180	1052	1022	1.029
8	1	2	356	2108	2046	1.030
9	2	0	760	4504	4094	1.100
10	2	1	1512	9016	8190	1.101
11	2	2	3016	18040	16382	1.101
22	3	8				1.1235
23	4	0				1.124994

Using different choices of a_1 and a_2 (found by computer), Theorem 1.1 can do somewhat better for $\mathrm{TC}_3(P^{3\cdot 2^e})$ than Theorem 1.4, but it does not seem worthwhile to try to find the best result implied by Theorem 1.1 for all e, since no pattern is apparent. For e from 7 to 11, the lower bounds for $\mathrm{TC}_3(P^{3\cdot 2^e})$ implied by Theorem 1.1 are, respectively, 1072, 2224, 4516, 9068, and 18284. For example, when $e = 11$, it is about 1.4% better than that implied by Theorem 3.3 and 11.6% better than that implied by cohomology. For one who wishes to check this result when $e = 11$, use $m = 3066$, $r = 3$, and $a_1 = 3287$ in Theorem 1.1. The values of $\nu\binom{a_1}{m-2^{r+\varepsilon}}$ (resp. $\nu\binom{a_2}{m-2^{r+\varepsilon}}$) for $\varepsilon = 0, 1, 2$ are (5,6,7) (resp. (6,6,3)).

4. $\mathrm{TC}_k(P^n)$ result implied by mod-2 cohomology, in a range

In this section, we prove that the lower bound for $\mathrm{TC}_k(P^n)$ implied by cohomology is constant in the last $\frac{2}{k}$ portion of the interval between successive 2-powers. This generalizes the behavior seen in Table 1 ($k = 3$) or Table 2 ($k = 4$) . In the previous section, we showed that the bound implied by BP rises in this range to a value nearly $k/(k-1)$ times that of the cohomology bound, which is as much as it possibly could.

Recall from [2] or [6] that $\mathrm{zcl}_k(P^n)$ is the lower bound for $\mathrm{TC}_k(P^n)$ implied by mod-2 cohomology. It is an analogue of Theorem 2.1, except that classes are in grading 1 rather than grading 2. Here we prove the following new result about $\mathrm{zcl}_k(P^n)$.

Theorem 4.1. *For $k \geqslant 3$ and $e \geqslant 2$, $\mathrm{zcl}_k(P^n) = (k-1)(2^e-1)$ for $[\frac{k-1}{k} \cdot 2^e] \leqslant n \leqslant 2^e - 1$.*

Note that, since $(k-1)n \leqslant \mathrm{zcl}_k(P^n) \leqslant kn$ (by [2] or [6]), this interval of constant $\mathrm{zcl}_k(P^n)$ is as long as it could possibly be.

Proof. We rely on [6, Thm 1.2], which can be interpreted to say that, with n_t denoting n mod 2^t,

$$\mathrm{zcl}_k(P^n) = kn - \max(2^{\nu(n+1)} - 1, kn_t - (k-1)(2^t-1)), \tag{4.2}$$

with the max taken over all t for which the initial bits of n mod 2^t begin a string of at least two consecutive 1's. That $\mathrm{zcl}_k(P^{2^e-1}) = (k-1)(2^e-1)$ is immediate from (4.2). Since $\mathrm{zcl}_k(P^n)$ is an increasing function of n, it suffices to prove

$$\text{if } n = [\tfrac{k-1}{k} \cdot 2^e], \text{ then } \mathrm{zcl}_k(P^n) = (k-1)(2^e-1). \tag{4.3}$$

The case $k = 3$ is slightly special since the binary expansion of $n = [2^{e+1}/3]$ does not have any consecutive 1's. For this n, (4.2) implies that $\mathrm{zcl}_3(P^n) = 3n + 1 - 2^{\nu(n+1)} = 2^{e+1} - 2$, as desired. From now on, we assume $k > 3$ in this proof.

One part that we must prove is

$$kn - 2^{\nu(n+1)} + 1 \geqslant (k-1)(2^e-1) \tag{4.4}$$

if n is as in (4.3). Write $2^e = Ak - \delta$ with $0 \leqslant \delta \leqslant k-1$. Then $n = 2^e - A$, and the desired inequality reduces to $k - \delta \geqslant 2^{\nu(A-1)}$ since $\nu(A-1) = \nu(2^e - A + 1)$. If $A - 1 = 2^t u$ with u odd, then $k - \delta = 2^e - 2^t uk \geqslant 2^t$ since $k - \delta > 0$, proving the inequality.

The rest of the proof requires the following lemma.

Lemma 4.5. *Let k be odd, and $\mathbf{e}$ the multiplicative order of 2 mod k. Thus $\mathbf{e}$ is the smallest positive integer such that k divides $2^{\mathbf{e}} - 1$. Let $m = (k-1)\frac{2^{\mathbf{e}}-1}{k}$, and let B be the binary expansion of m. If $t = \alpha\mathbf{e} + \beta$ with $0 \leqslant \beta < \mathbf{e}$, then the binary expansion of $[(k-1)2^t/k]$ consists of the concatenation of α copies of B, followed by the first β*

bits of B. Also, the binary expansion of $[(2^v k-1)2^{v+t}/(2^v k)]$ with k odd equals that of $[(k-1)2^t/k]$ preceded by v 1's. If $k \geqslant 4$, B begins with at least two 1's.

Proof. Let $f_t=(k-1)2^t/k$. Then, letting $\{f\}=f-[f]$ denote the fractional part of f,

$$[f_{t+1}]=\begin{cases}2[f_t] & \text{if } \{f_t\}<1/2\\ 2[f_t]+1 & \text{if } \{f_t\}\geqslant 1/2.\end{cases}$$

This shows that as t increases, the binary expansions of the $[f_t]$ are just initial sections of subsequent ones. They start with at least two 1's when $k \geqslant 4$ since $[2^2(k-1)/k]=3$.

If $\mathbf{e}$ is as in the lemma, then

$$\frac{(k-1)2^{t+\mathbf{e}}}{k}-\frac{(k-1)2^t}{k}=2^t\frac{(k-1)(2^{\mathbf{e}}-1)}{k},$$

showing that adding this $\mathbf{e}$ to the exponent just appends B in front of the binary expansion. Regarding $2^v k$, note that

$$\frac{(2^v k-1)2^{t+v}}{2^v k}=(2^v-1)2^t+\frac{(k-1)2^t}{k},$$

which shows the appending of 1's in front. ■

In Table 4, we list some values of B, the binary expansion of m, for the m associated to k as in Lemma 4.5.

Table 4. Binary expansions B of numbers appearing in lemma

k	$\mathbf{e}$	B
9	6	111000
11	10	1110100010
13	12	111011000100
15	4	1110
17	8	11110000
19	18	111100101000011010
21	6	111100
23	11	11110100110

The property (4.7) says roughly that the beginning of B has more 1's than anywhere else in B.

For any $k>3$ and $n=[\frac{k-1}{k}\cdot 2^e]$ as in (4.3), equations (4.2) and (4.4) imply that

$$\mathrm{zcl}_k(P^n)\leqslant kn-(kn-(k-1)(2^e-1))=(k-1)(2^e-1),$$

with equality if, for all t for which the initial bits of n mod 2^t begin a string of at least two consecutive 1's,

$$kn_t - (k-1)(2^t - 1) \leqslant kn - (k-1)(2^e - 1).$$

This is equivalent to

$$1 - \tfrac{1}{k} \leqslant \tfrac{n-n_t}{2^e - 2^t}. \tag{4.6}$$

By the lemma, if k is odd (resp. even), the RHS of (4.6) is the same as (resp. greater than) it would be if (n, e) is replaced by $(m, \mathbf{e})$, with notation as in the lemma, provided $t \leqslant \mathbf{e}$. Note that equality holds in (4.6) if (n, e, t) is replaced by $(m, \mathbf{e}, 0)$. Hence, again using the lemma for cases in which $t > \mathbf{e}$, (4.6) will follow from its validity if (n, e) is replaced by $(m, \mathbf{e})$, and, since $1 - \frac{1}{k} = \frac{m}{2^{\mathbf{e}}-1}$, this reduces to showing

$$\tfrac{m_t}{2^t - 1} \leqslant \tfrac{m}{2^{\mathbf{e}} - 1}. \tag{4.7}$$

Let $q = \frac{2^{\mathbf{e}}-1}{k} = 2^{\mathbf{e}} - 1 - m$ and $q_t = 2^t - 1 - m_t$ its reduction mod 2^t. Now the desired inequality reduces to $\frac{q_t}{2^t-1} \geqslant \frac{q}{2^{\mathbf{e}}-1} = \frac{1}{k}$; i.e., $kq_t \geqslant 2^t - 1$. We can prove the validity of this last inequality as follows. Write $q = q_t + 2^t\alpha$, for an integer α. Then

$$2^{\mathbf{e}} - 1 = kq = kq_t + 2^t\alpha k.$$

Reducing mod 2^t gives the desired result. ■

Remark 4.8. It appears that the stronger inequality $kq_t \geqslant 3 \cdot 2^t - 1$ holds when $q = \frac{2^{\mathbf{e}}-1}{k}$, but we do not need it, and it seems much harder to prove.

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