BOUNDS FOR HIGHER TOPOLOGICAL COMPLEXITY OF REAL PROJECTIVE SPACE IMPLIED BY BP

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Abstract. We use BP to obtain lower bounds for $\mathrm{TC}_3(RP^{2m})$.

1. Introduction and results

In [2], the study of the higher (reduced) topological complexity of real projective spaces, $\mathrm{TC}_s(RP^n)$, was initiated, and some lower bounds were obtained using mod-2 cohomology. Since $\mathrm{TC}_2(RP^n)$ is usually equal to the immersion dimension([6]), and a sweeping family of strong nonimmersion results was obtained using the Brown-Peterson spectrum BP in [3], one is led to apply BP to obtain lower bounds for $\mathrm{TC}_s(P^n)$ for $s > 2$. Here we initiate this study by obtaining a family of lower bounds for $\mathrm{TC}_3(RP^{2m})$ for many values of m which are often much stronger than those implied by cohomology. We will discuss in Section 4 how there are more results to be obtained by these methods, but the results presented here seem to be the simplest broad family.

Here is our main result. We write $\nu(k)$ for the exponent of 2 in an integer k, and $\lg(n) = [\log_2(n)]$ with $\lg(0) = -1$.

Theorem 1.1. *If $r \geqslant 0$, $t \geqslant 2$, and $m = 2^{t+r}(4n+3) + \Delta$ with $2^r \leqslant \Delta < \frac{1}{3}(2^{t+r} + 2^{r+2} - 1)$, let*

$$C_n := \binom{6n - 2^{\lg(n)+2} + 4}{2n+1} \text{ and } C_{\Delta,r} := \begin{cases} \binom{3\Delta+1-2^{r+2}}{\Delta-2^{r+1}} & \Delta \geqslant 2^{r+1} \\ \binom{3\Delta+1}{\Delta} & 2^r \leqslant \Delta < 2^{r+1}. \end{cases}$$

If $\nu(C_n) = 2^r$ and $\nu(C_{\Delta,r}) = 0$, then $\mathrm{TC}_3(RP^{2m}) \geqslant 6m - 14 \cdot 2^r$.

A more intuitive description of some of these conditions can be given in terms of binary expansions. Recall that Fibbinary numbers are those having no adjacent 1's

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in their binary expansion; equivalently, they are the numbers n such that $\binom{3n}{n}$ is odd. The Fibbinary numbers arise here, as well as others that we call "almost Fibbinary." Let $\varepsilon_j(n)$ denote the entries in the binary expansion of n, so that $n = \sum \varepsilon_j(n)2^j$.

Definition 1.2. *For a nonnegative integer e, AF_e is the set of positive integers n such that $\varepsilon_e(n) = 1$, and if $j \neq e$, then $\varepsilon_j(n) + \varepsilon_{j+1}(n) < 2$. (Note that $\varepsilon_{e+1}(n)$ may equal either 0 or 1.)*

In Section 2, we will prove the following results, involving notation from Theorem 1.1.

Proposition 1.3. *$\nu(C_n) = 1$ iff $n = 0$, while $\nu(C_n) = 2$ iff $n = 1$ or n is a Fibbinary number divisible by 4.*

Proposition 1.4. *For $r \geqslant 0$, $\nu(C_{\Delta,r}) = 0$ if and only if Δ is even and*

$$\begin{cases} \Delta \in \mathrm{AF}_{r-1} \cup \mathrm{AF}_r & \Delta \geqslant 2^{r+1} \\ \Delta \textit{ is Fibbinary} & 2^r \leqslant \Delta < 2^{r+1}. \end{cases}$$

There are no even numbers in AF_0, while the first few even numbers in AF_1 are 2, 6, 10, 18, and 22, and the even numbers in AF_2 are just twice those in AF_1. Using this, we easily see that the first few numbers m for which Theorem 1.1 implies $\mathrm{TC}_3(RP^{2m}) \geqslant 6m - 14$ are

$$14, 26, 50, 54, 98, 102, 106, 194, 198, 202, 210, 214,$$

and the first few for which Theorem 1.1 implies $\mathrm{TC}_3(RP^{2m}) \geqslant 6m - 28$ are

$$58, 60, 114, 116, 118, 154, 156, 226, 228, 230, 234, 236.$$

These results are all stronger than the results implied by cohomology, as we will discuss more thoroughly in Section 3, where we give an interesting description of all bounds implied by cohomology. For example, when $m = 236$, we prove that $\mathrm{TC}_3(P^{2m}) \geqslant 1388$, while cohomology only gives $\mathrm{TC}_3(P^{2m}) \geqslant 1022$. The smallest m for which Theorem 1.1 applies with $r = 2$ (resp. $r = 3$) is 372 (resp. 10984).

2. Proofs

The first step of our approach follows suggestions of Jesus González, and is similar to work in [2]. We write it here for TC_3; there is an obvious adaptation to TC_s, which we hope to consider in the future. Let P^n denote the real projective space. There are canonical elements X_1, X_2, and X_3 in $BP^2(P^n \times P^n \times P^n)$.

Theorem 2.1. *If* $(X_1 - X_2)^a(X_2 - X_3)^b \neq 0 \in BP^*(P^n \times P^n \times P^n)$*, then* $\mathrm{TC}_3(P^n) \geqslant 2a + 2b$*.*

Proof. Let $(P^n)^{[0,1]}$ denote the space of paths in P^n, and

$$P_{(n,n,n)} = (S^n \times S^n \times S^n)/((z_1, z_2, z_3) \sim (-z_1, -z_2, -z_3))$$

the projective product space.([5]) There is a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-cover $P_{(n,n,n)} \xrightarrow{\pi} P^n \times P^n \times P^n$, which is classified by a map $P^n \times P^n \times P^n \xrightarrow{\mu} P^\infty \times P^\infty = B(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The map $(P^n)^{[0,1]} \xrightarrow{p} P^n \times P^n \times P^n$ defined by $\sigma \mapsto (\sigma(0), \sigma(\frac{1}{2}), \sigma(1))$ lifts to a map $(P^n)^{[0,1]} \xrightarrow{\widetilde{p}} P_{(n,n,n)}$. A definition of $\mathrm{TC}_3(P^n)$ is as the sectional category $\mathrm{secat}(p)$. The lifting $\widetilde{p}$ implies that $\mathrm{secat}(p) \geqslant \mathrm{secat}(\pi)$.

Let $B_k(\mathbb{Z}_2 \times \mathbb{Z}_2) = \ast^{k+1}(\mathbb{Z}_2 \times \mathbb{Z}_2)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, where $\ast^{k+1}$ denotes the $(k+1)$-fold iterated join; this is the kth stage in Milnor's construction of $B(\mathbb{Z}_2 \times \mathbb{Z}_2)$, with a map $i_k : B_k(\mathbb{Z}_2 \times \mathbb{Z}_2) \to B(\mathbb{Z}_2 \times \mathbb{Z}_2)$. By [7, Thm 9, p. 86], as described in [2, §4], μ lifts to a map $P^n \times P^n \times P^n \xrightarrow{\widetilde{\mu}} B_{\mathrm{secat}(\pi)}(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

$$\begin{tikzcd}
(P^n)^{[0,1]} \arrow[r, "\widetilde{p}"] \arrow[dr, "p"'] & P_{(n,n,n)} \arrow[d, "\pi"] & & B_k(\mathbb{Z}_2 \times \mathbb{Z}_2) \arrow[d, "i_k"] \\
& P^n \times P^n \times P^n \arrow[urr, "\widetilde{\mu}"] \arrow[rr, "\mu"] & & B(\mathbb{Z}_2 \times \mathbb{Z}_2)
\end{tikzcd}$$

By [2, Prop 3.1], μ classifies $(p_1^*(\xi) \otimes p_3^*(\xi)) \times (p_2^*(\xi) \otimes p_3^*(\xi))$, and so, by [1, Prop 3.6], the induced homomorphism

$$BP^*(P^\infty \times P^\infty) \xrightarrow{\mu^*} BP^*(P^n \times P^n \times P^n)$$

satisfies $\mu^*(X_i) = u_i(X_i - X_3)$ for $i \in \{1, 2\}$, with u_i a unit. Since $\mu^* = \widetilde{\mu}^* i_k^*$ and $B_k(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is k-dimensional, $\mu^*(X_1^a X_2^b) = 0$ if $2a + 2b > k$, implying the theorem. ■

Let I denote the ideal $(v_0, v_1, v_2, v_3) \subset BP^*$. Recall $v_0 = 2$ and $|v_3| = 14$. In $BP^*(X)$, let F_s denote the BP^*-submodule $I^s \cdot BP^*(X)$. Let $\mathcal{P}(s_1, s_2, s_3)$ denote the set consisting of the six permutations of distinct integers s_1, s_2, and s_3. In [4, Cor

2.4], it is proved that in $BP^*(P^{2m}\times P^{2m}\times P^{2m})$, for $r\geqslant 0$ and integers ℓ_1, ℓ_2, and ℓ_3,

$$2^{2^r}X_1^{\ell_1}X_2^{\ell_2}X_3^{\ell_3}\equiv v_3^{2^r}\sum X_1^{\ell_1+i}X_2^{\ell_2+j}X_3^{\ell_3+k}\mod F_{2^r+1},\tag{2.2}$$

where the sum is taken over all (i,j,k) in $\mathcal{P}(2^r,2^{r+1},2^{r+2})$. Some similar results were obtained in [9], where it was noted that by the (proven) Conner-Floyd conjecture, $v_3^dX_1^mX_2^mX_3^m\neq 0$ for any nonnegative integer d. We use these observations to deduce the following result.

Proposition 2.3. *Let $r\geqslant 0$, and suppose $a+b=3m-7\cdot 2^r$ and $\nu\bigl(\binom{a}{i}\binom{b}{j}\bigr)\geqslant 2^r$ for all $i,j\leqslant m$ which satisfy also that $i+j\geqslant 2m-7\cdot 2^r$. Suppose also that an odd number of the numbers $\nu\bigl(\binom{a}{m-k}\binom{b}{m-k'}\bigr)$ are equal to 2^r, as (k,k') ranges over the six ordered pairs in $\{2^r,2^{r+1},2^{r+2}\}$. Then $\mathrm{TC}_3(P^{2m})\geqslant 6m-14\cdot 2^r$.*

Proof. Let $h=2^r$ in this proof. The result follows from Theorem 2.1 once we show that

$$(X_1-X_3)^a(X_2-X_3)^b\neq 0\in BP^{6m-14h}(P^{2m}\times P^{2m}\times P^{2m}).$$

This expands as $\sum\limits_{i,j}\pm\binom{a}{i}\binom{b}{j}X_1^iX_2^jX_3^{3m-7h-i-j}$, for values of i and j described in the proposition. By (2.2), this equals, mod F_{h+1},

$$v_3^h\sum_{i,j}\sum_{k,k'}\pm 2^{-h}\binom{a}{i}\binom{b}{j}X_1^{i+k}X_2^{j+k'}X_3^{3m-i-j-k-k'}.$$

These terms are 0 unless $i=m-k$ and $j=m-k'$, since they would contain a factor X^ℓ with $\ell>m$. We are left with

$$\Bigl(\sum_{k,k'}\pm 2^{-h}\binom{a}{m-k}\binom{b}{m-k'}\Bigr)v_3^hX_1^mX_2^mX_3^m$$

with (k,k') as above, and this is nonzero by the hypothesis and the remark just preceding the theorem. ■

Theorem 1.1 is an almost immediate consequence of Proposition 2.3 and the following combinatorial result.

Theorem 2.4. *With m, r, t, n, and Δ as in Theorem 1.1, let $a=2^{t+r+\lg(n)+3}-2^r-1$ and $a+b=3m-7\cdot 2^r$. Then*

a. *If* $m - 7\cdot 2^r \leqslant j \leqslant m$, *then* $\nu\binom{b}{j} \geqslant 2^r$.
b. *Mod* 2, $\binom{a}{m-2^{r+1}} \equiv \binom{a}{m-2^{r+2}} \not\equiv \binom{a}{m-2^r}$.
c. $\nu\left(\binom{b}{m-2^{r+1}} + \binom{b}{m-2^{r+2}}\right) = 2^r$.

To see that this implies the hypotheses of Proposition 2.3 and hence Theorem 1.1, one needs just to check that the conditions on i and j in 2.3 imply the range for j in (a.), and that (b.) and (c.) imply the hypothesis in 2.3 about the divisibility of the six numbers $\binom{a}{m-k}\binom{b}{m-k'}$.

We will often use without comment Lucas's Theorem regarding binomial coefficients mod 2, and that

$$\nu\binom{m}{n} = \alpha(n) + \alpha(m-n) - \alpha(m),$$

where $\alpha(-)$ denotes the number of 1's in the binary expansion.

Proof of 2.4.a. Write $b = 2^{t+r}B_1 + B_2$ and $j = 2^{t+r}J_1 + J_2$, with

$$B_1 = 12n - 2^{\lg(n)+3} + 8, \qquad B_2 = 3\Delta + 1 + (2^t - 6)2^r,$$
$$J_1 = 4n + 2, \qquad J_2 = 2^{t+r} + \Delta - d \text{ with } 0 \leqslant d \leqslant 7\cdot 2^r.$$

The upper bound on Δ implies that $B_2 < 2^{t+r+2}$ and $J_2 < 2^{t+r+1}$, so the binary expansions of B_1 and J_1 split as above if $J_2 \geqslant 0$ (since we also have $B_2 \geqslant 0$), and hence $\nu\binom{b}{j} \geqslant \nu\binom{B_1}{J_1} \geqslant 2^r$, by assumption. This will be the case if $t > 2$.

If $J_2 < 0$ (whence $t = 2$ and $d > 5\cdot 2^r$), write j as $2^{2+r}(4n+1) + J_2'$ with $J_2' = (2^{3+r} + \Delta - d) > 0$, and note that $\alpha(j) = \alpha(J_1) + \alpha(J_2')$. Then

$$\begin{aligned}\nu\binom{b}{j} &= \nu\binom{B_1}{J_1} + \alpha(2^{3+r} + \Delta - d) + \alpha(2\Delta + 1 + d - 6\cdot 2^r) - \alpha(3\Delta + 1 - 2^{r+1})\\ &= \nu\binom{B_1}{J_1} + \nu\binom{3\Delta+1+2^{r+1}}{2^{3+r}+\Delta-d} + \alpha(3\Delta + 1 + 2^{r+1}) - \alpha(3\Delta + 1 - 2^{r+1})\\ &\geqslant 2^r,\end{aligned}$$

since $3\Delta + 1 - 2^{r+1} < 3\cdot 2^{r+2}$, so adding 2^{r+2} to it cannot decrease its $\alpha(-)$. ■

Proof of 2.4.b. The mod 2 value of $\binom{a}{\ell}$ equals $1 - \varepsilon_r(\ell)$, provided $\ell \leqslant a$. Part (b) follows immediately. ■

Proof of 2.4.c. Write $b = 2^{t+r}B_1 + B_2$ as in the proof of part a, and, for $\varepsilon \in \{1,2\}$, write $m - 2^{r+\varepsilon} = 2^{t+r}(4n+2) + M_\varepsilon$ with $M_\varepsilon = \Delta - 2^{r+\varepsilon} + 2^{t+r}$. We have $0 \leqslant M_\varepsilon < 2^{t+r+1}$, so

$\nu\binom{b}{m-2^{r+\varepsilon}} \geqslant 2^r$ with equality iff $\binom{B_2}{M_\varepsilon}$ is odd. Thus $\nu\left(\binom{b}{m-2^{r+1}}+\binom{b}{m-2^{r+2}}\right) \geqslant 2^r$ with equality iff $\binom{B_2}{M_1}+\binom{B_2}{M_2}$ is odd. Mod 2, by iterating Pascal's formula, this latter sum is $\binom{B_2+2^{r+1}}{M_1}=\binom{3\Delta+1-2^{r+2}+2^{t+r}}{\Delta-2^{r+1}+2^{t+r}}$. If $\Delta \geqslant 2^{r+1}$, then both $3\Delta+1-2^{r+2}$ and $\Delta-2^{r+1}$ lie in the interval $[0,2^{t+r})$, and so this binomial coefficient, mod 2, equals $\binom{3\Delta+1-2^{r+2}}{\Delta-2^{r+1}}$, which is odd by hypothesis.

If $2^r \leqslant \Delta < 2^{r+1}$, we claim that $\binom{3\Delta+1-2^{r+2}+2^{t+r}}{\Delta-2^{r+1}+2^{t+r}}$ is odd iff $\binom{3\Delta+1}{\Delta}$ is odd. If $\binom{3\Delta+1}{\Delta}$ is even, it is due to a 0 over 1 in some position less than 2^{r+1}, and the added terms do not affect that in the other binomial coefficient. If $\binom{3\Delta+1}{\Delta}$ is odd, then Δ is a Fibbinary number, and so $2^{r+1} \leqslant 3\Delta+1 < 2^{r+2}$. Thus $\varepsilon_i(3\Delta+1-2^{r+2}+2^{t+r}) = 1 = \varepsilon_i(\Delta-2^{r+1}+2^{t+r})$ for $r+1 \leqslant i < t+r$, and earlier positions will be the same as $3\Delta+1$ and Δ. Thus $\binom{3\Delta+1-2^{r+2}+2^{t+r}}{\Delta-2^{r+1}+2^{t+r}}$ is odd. ■

Finally, we prove Propositions 1.3 and 1.4.

Proof of Proposition 1.3. We easily check the exceptional cases $n = 0$ and $n = 1$, and write $n = 2^e + d$ with $0 \leqslant d < 2^e$. Let $\mathcal{F}$ denote the set of Fibbinary numbers. Note that $n \in \mathcal{F}$ iff $d \in \mathcal{F}$ and $3d < 2^e$, since the largest $n \in \mathcal{F}$ is $2^e + 2^{e-2} + \cdots$. We study

$$\begin{aligned} V &= \nu\binom{6n-2^{\lg(n)+2}+4}{2n+1} \\ &= \nu\binom{2^{e+1}+6d+4}{2^{e+1}+2d+1} \\ &= \alpha(2^{e+1}+2d+1)+\alpha(4d+3)-\alpha(2^e+3d+2) \\ &= 2\alpha(d)+4-\alpha(2^e+3d+2). \end{aligned}$$

If $3d+2 < 2^e$ or $3d+2 \geqslant 2^{e+1}$, then $V = 2\alpha(d)+3-\alpha(3d+2)$. Since

$$\alpha(3d+2)\begin{cases} = \alpha(3d)+1 & d \equiv 0,3\ (4) \\ < \alpha(3d)+1 & d \equiv 1,2\ (4), \end{cases}$$

we obtain that for these d,

$$V\begin{cases} = \nu\binom{3d}{d}+2 & d \equiv 0,3\ (4) \\ > \nu\binom{3d}{d}+2 & d \equiv 1,2\ (4). \end{cases}$$

Since n has adjacent 1's if $d \equiv 3\ (4)$ or if $3d \geqslant 2^{e+1}$, this proves the claim for $3d+2 < 2^e$ or $3d+2 \geqslant 2^{e+1}$.

If $2^e \leqslant 3d+2 < 2^{e+1}$, then $\alpha(2^e+3d+2) = \alpha(3d+2)$, so V is 1 larger than in the above cases; hence $V \geqslant 3$. By the earlier "Note," $n \notin \mathcal{F}$ for d in this range, completing the proof. ■

Proof of Proposition 1.4. The second case is elementary, so we focus on the first. Let P_r denote the statement that for $n \geqslant 2^r$, $\binom{3n+1-2^{r+1}}{n-2^r}$ is odd iff n is even and $n \in \mathrm{AF}_{r-1} \cup \mathrm{AF}_r$, and let Q_r denote the statement that $\binom{3n-2^{r+1}}{n-2^r}$ is odd iff $n \in \mathrm{AF}_{r-1} \cup \mathrm{AF}_r$. We prove both P_r and Q_r by induction on r. It is easy to see that, for $r \geqslant 2$, Q_{r-1} implies P_r, and that Q_{r-1} and P_{r-1} imply Q_r. We prove the second. The first is similar and easier.

If $n = 2n'$, then $\binom{3n-2^{r+1}}{n-2^r} \equiv \binom{3n'-2^r}{n'-2^{r-1}}$. By Q_{r-1}, this is odd iff $n' \in \mathrm{AF}_{r-2} \cup \mathrm{AF}_{r-1}$ iff $n \in \mathrm{AF}_{r-1} \cup \mathrm{AF}_r$. If $n = 2n'+1$, then $\binom{3n-2^{r+1}}{n-2^r} \equiv \binom{3n'+1-2^r}{n'-2^{r-1}}$. By P_{r-1}, this is odd iff n' is even and $n' \in \mathrm{AF}_{r-2} \cup \mathrm{AF}_{r-1}$, and this is true iff $n \in \mathrm{AF}_{r-1} \cup \mathrm{AF}_r$. Note that if n' is odd, then $n \equiv 3\ (4)$, so $n \notin \mathrm{AF}_{r-1} \cup \mathrm{AF}_r$.

It remains to prove P_1 and Q_1. We prove Q_1. Proving P_1 is similar and slightly easier. We must prove that for $n \geqslant 2$, $\binom{3n-4}{n-2}$ is odd iff n is odd and $\binom{3(n-1)}{n-1}$ is odd, or $n \equiv 2\ (4)$ and $\binom{3(n-2)}{n-2}$ is odd. If $n = 2n'+1$, we need that $\binom{6n'-1}{2n'-1} \equiv \binom{6n'}{2n'}$, which is true since they differ by a factor $\frac{6n'}{2n'}$. If $n \equiv 0\ (4)$, $\binom{3n-4}{n-2}$ is even. If $n = 4n'+2$, then the result follows from the true statement $\binom{12n'+2}{4n'} \equiv \binom{12n'}{4n'}$. ■

3. Results obtained using mod-2 cohomology

In [2], some lower bounds obtainable using mod-2 cohomology were discussed, but complete results were not obtained. Here we obtain the complete result for lower bounds for $TC_3(RP^n)$ obtainable using $\mathbb{Z}_2$-cohomology, which involve an interesting new family of numbers. The following result was noted in [2].

Proposition 3.1. *Let $Z(n)$ denote the largest integer s such that there exists an integer a with $(X_1+X_3)^a(X_2+X_3)^{s-a} \neq 0 \in H^*(P^n \times P^n \times P^n; \mathbb{Z}_2)$. Then $\mathrm{TC}_3(P^n) \geqslant Z(n)$.*

We determine $Z(n)$ in terms of some new numbers z_n, which we now define.

Definition 3.2. *Let z_n denote the largest number $z \leqslant 3n+1$ such that $\binom{z}{n}$ is odd.*

The first few values of z_n are given in Table 1.

TABLE 1. Values of z_n

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
z_n	1	3	7	7	13	15	15	15	25	27	31	31	31	31	31	31	49	51	55	55

A formula for z_n can be given in terms of the binary expansion of n.

Proposition 3.3. *Let $\mathcal{F}$ denote the set of Fibbinary numbers (those n with $\binom{3n}{n}$ odd). If $n \in \mathcal{F}$, then $z_n = 3n + 1$ if n is even, and $3n$ if n is odd. Otherwise, let e be the largest integer such that $\varepsilon_e(n) = 1 = \varepsilon_{e-1}(n)$, so that $n = A + B$, with $A \in \mathcal{F}$ and A divisible by 2^{e+2} and $3 \cdot 2^{e-1} \leqslant B < 2^{e+1}$. Then $z_n = 3A + 2^{e+2} - 1$.*

Proof. It is clear when $n \in \mathcal{F}$. If $n = A + B$ and $w = 3A + 2^{e+2} - 1$, clearly $\binom{w}{n}$ is odd, and $w < 3n + 1$ since $2^{e+2} - 1 < 3 \cdot 3 \cdot 2^{e-1}$. Any v larger than w and $\leqslant 3n + 1$ will have $(v \mod 2^{e+1}) < (n \mod 2^{e+1})$ and hence $\binom{v}{n}$ even. To see this, write $B = 2^e + 2^{e-1} + \ell$ with $\ell < 2^{e-1}$. Then $(v \mod 2^{e+1}) \leqslant 2^{e-1} + 3\ell + 1$ while $(n \mod 2^{e+1}) = 2^e + 2^{e-1} + \ell$. The result follows since $2\ell + 1 < 2^e$. ■

The relevance of z_n is given by the following result, involving the numbers $Z(n)$ of Proposition 3.1.

Theorem 3.4. *If $n = 2^e + d$ with $0 \leqslant d < 2^e$, then $Z(n) = 3 \cdot 2^e - 1 + \min(z_d, 2^e - 1)$.*

Proof. First note that $Z(n)$ is the largest $a + b$ such that there exist integers $i, j \leqslant n$ such that $\binom{a}{i}\binom{b}{j}$ is odd and $i + j \geqslant a + b - n$. With $m = \min(z_d, 2^e - 1)$, setting $a = 2^{e+1} - 1$, $b = 2^e + m$, and $i = j = n$ shows that $3 \cdot 2^e - 1 + m \leqslant Z(n)$. ⟦The condition $i + j \geqslant a + b - n$ follows from $3d + 1 \geqslant m$.⟧

We now show that $a + b$ cannot be increased from the above values. If a is increased to a number $\geqslant 2^{e+1}$, then $\binom{a}{i}$ odd implies $\binom{a-2^{e+1}}{i}$ odd. Thus

$$a + b - n \leqslant i + j \leqslant (a - 2^{e+1}) + b,$$

contradicting $n < 2^{e+1}$. On the other hand, if b is increased to $2^e + k$ with $k > z_d$, and $j = 2^e + \ell$ with $\ell \leqslant d$, then $k \leqslant 2d + \ell + 1$. ⟦This follows from $a + b - n \leqslant i + j \leqslant n + j$.⟧ By definition of z_d, we have $\binom{t}{d}$ even for $z_d < t \leqslant 3d + 1$. Hence Pascal's formula implies inductively that $\binom{k}{d-\delta}$ is even for $z_d < k \leqslant 3d+1-\delta$. Letting $\ell = d-\delta$, we obtain that $\binom{k}{\ell}$ is even for k and ℓ as above, and hence so is $\binom{b}{j}$. ∎

Values of $Z(2m)$ for $32 \leqslant m \leqslant 63$ appear in the second column of Table 2. These values for $m = 32+d$ are $191 + \min(z_{2d}, 63)$.

4. Obtaining more results

In this section, we discuss briefly how we might obtain additional results using BP. One obvious thing to do is to study products of more than three copies of RP^{2m}; i.e., $\mathrm{TC}_s(RP^{2m})$ for $s > 3$. Our approach should extend without significant change. For $\mathrm{TC}_3(-)$ itself, there are three other things that we might try.

One way in which we might expand our results is by considering non-2-power divisibilities for our obstructions. In [4], there is a result which implies an analogue of our Proposition 2.3 when 2^r is replaced by an arbitrary positive integer h. It is more complicated with respect to the pairs (k, k') which must be considered there, and so we have not pursued it.

Another way, which we would hope to pursue, is to use values of a in Proposition 2.3 other than the $2^{\lg(m)+1} - 2^r - 1$ which we used in proving Theorem 1.1. For $r = 0$ and 1, we have run `Maple` programs to find all $m < 256$ and a for which the hypothesis of Proposition 2.3 holds. There are many, but finding generalizable patterns to prove is not an easy task. One that apparently works for $r = 0$, but does not seem to have an analogue for larger r, is that for all the values of m for which Theorem 1.1 applied with $r = 0$, using $a = 2^{\lg(m)+1} - 2$, we can use $a = 2^{\lg(m)+1} - 3$ for $m' = m-1$ to prove $\mathrm{TC}_3(RP^{2m'}) \geqslant 6m' - 14$. In addition to the values of m for which Theorem 1.1 implies $\mathrm{TC}_3(RP^{2m}) \geqslant 6m - 14$, which were listed in Section 1, and the values of m one less than these, the following values of $m < 256$ admit values of a which enable us to deduce this conclusion.

$$10, 18, 21, 22, 34, 37, 38, 41, 42, 66, 69, 70, 73, 74, 81, 82, 85, 86, 101, 102, 105,$$
$$106, 130, 133, 134, 137, 138, 145, 146, 149, 150, 161, 162, 165, 166, 169, 170$$

Similarly there are 36 values of $m < 256$ for which we can use $r = 1$ in Proposition 2.3 to prove $\mathrm{TC}_3(RP^{2m}) \geqslant 6m - 28$, in addition to those implied by Theorem 1.1.

For $32 \leqslant m \leqslant 63$, we list in Table 2 the lower bounds for $\mathrm{TC}_3(P^{2m})$ implied by $H^*(-;\mathbb{Z}_2)$ and by BP with $r=0$ and 1, using all values of a, not just the one used in Theorem 1.1. Note that cohomology gives a better estimate in the first 17 cases, and BP in the last 15.

Another way of obtaining more results would be to consider relations in $BP^*(RP^n \times RP^n \times RP^n)$ totally different than those used in Proposition 2.3. In [3], all relations in $BP^{2*}(RP^n \times RP^n)$ were obtained and used. This seems daunting for larger products.

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TABLE 2. Lower bounds for $\mathrm{TC}_3(P^{2m})$ implied by $H^*(-)$ and BP

m	$H^*(-)$	$r=0$	$r=1$
32	192	142	152
33	198	142	152
34	204	190	152
35	206	190	152
36	216	190	152
37	222	208	152
38	222	214	152
39	222	214	152
40	240	214	152
41	246	232	152
42	252	238	152
43	254	238	152
44	254	238	236
45	254	238	236
46	254	238	248
47	254	238	248
48	254	238	248
49	254	280	248
50	254	286	248
51	254	286	248
52	254	286	284
53	254	304	284
54	254	310	296
55	254	310	296
56	254	310	296
57	254	310	296
58	254	310	320
59	254	310	320
60	254	310	332
61	254	310	332
62	254	310	332
63	254	310	332