BP-HOMOLOGY AND AN IMPLICATION FOR SYMMETRIC POLYNOMIALS

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Abstract. We determine the BP_*-module structure, mod higher filtration, of the main part of the BP-homology of elementary 2-groups. In doing this, we show, using algebraic topology, that certain Vandermonde-type systems of equations, mod 2, have solutions that are symmetric polynomials. We then show that in analogous situations not covered by our theorem, instead of polynomials, the solution involves infinite series containing many negative exponents. Equivalently, we have similar results regarding whether quotients of Schur polynomials are polynomials mod 2.

1. Introduction and results

Let $BP_*(-)$ denote Brown-Peterson homology localized at 2. Its coefficient groups BP_* are a polynomial algebra over $\mathbb{Z}_{(2)}$ on classes v_j, $j \geq 1$, of grading $2(2^j-1)$. Let $v_0 = 2$. There are BP_*-generators $z_i \in BP_{2i-1}(B\mathbb{Z}/2)$ for $i \geq 1$. As was done in [2] and [4], we consider $\bigotimes^k_{BP_*} BP_*(B\mathbb{Z}/2)$, which is a BP_*-direct summand of $BP_*(B(\mathbb{Z}/2)^k)$. This contains classes $z_I = z_{i_1} \otimes \cdots \otimes z_{i_k}$ for $I = (i_1, \ldots, i_k)$ with $i_j \geq 1$. Let Z_k denote the set consisting of all such classes z_I. It was proved in [2, Thm 3.2] that $\bigotimes^k_{BP_*} BP_*(B\mathbb{Z}/2)$ admits a decreasing filtration by BP_*-submodules F_s such that, for $s \geq 0$, the quotient F_s/F_{s+1} is a $\mathbb{Z}/2$-vector space with basis all classes $(v_k^{t_k} v_{k+1}^{t_{k+1}} \cdots) z_I$ with $z_I \in Z_k$ and $\sum t_i = s$.

Define an action of $\mathbb{Z}/2[x_1, \ldots, x_k]$ on the $\mathbb{Z}/2$-vector space with basis Z_k by $x_1^{e_1} \cdots x_k^{e_k} \cdot z_I = z_{I-E}$, where $I - E = (i_1 - e_1, \ldots, i_k - e_k)$. Note that $z_J = 0$ if any entry of J is ≤ 0.

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Our first theorem determines the action of v_j, $0 \le j \le k-1$, from F_s/F_{s+1} to F_{s+1}/F_{s+2}, obtained as the solution of a system of linear equations. This theorem and the next one will be proved in Section 2.

Theorem 1.1. *If F_s is as above, and $0 \le j \le k-1$, the action of v_j from F_s/F_{s+1} to F_{s+1}/F_{s+2} is multiplication by $\sum\limits_{\ell \ge k} v_\ell p_{j,k,\ell}$, where $p_{j,k,\ell}$ are mod-2 symmetric polynomials in $x_1, \dots, x_k$ satisfying the mod-2 system*

$$\begin{bmatrix} 1 & x_1 & x_1^3 & \cdots & x_1^{2^{k-1}-1} \\ & & \vdots & & \\ 1 & x_k & x_k^3 & \cdots & x_k^{2^{k-1}-1} \end{bmatrix} \begin{bmatrix} p_{0,k,\ell} \\ \vdots \\ p_{k-1,k,\ell} \end{bmatrix} = \begin{bmatrix} x_1^{2^\ell - 1} \\ \vdots \\ x_k^{2^\ell - 1} \end{bmatrix}. \tag{1.2}$$

It is not *a priori* clear that the system (1.2) should have a polynomial solution, mod 2. In fact, as we will show in Theorem 1.3, if the exponents on the RHS of (1.2) are replaced by an integer $\ge 2^{k-1}$ which is not of the form $2^\ell - 1$ for some ℓ, then (1.2) does not have a polynomial solution mod 2. Part of the content of Theorem 1.1 is that the system (1.2) has a polynomial solution mod 2, and the essence of the proof is algebraic topology.

Theorem 1.3. *Let $k \ge 3$ and $t \ge 2^{k-1}$. Let*

$$A = \begin{bmatrix} 1 & x_1 & x_1^3 & \cdots & x_1^{2^{k-1}-1} \\ & & \vdots & & \\ 1 & x_k & x_k^3 & \cdots & x_k^{2^{k-1}-1} \end{bmatrix} \text{ and } B = \begin{bmatrix} x_1^t \\ \vdots \\ x_k^t \end{bmatrix}.$$

The mod-2 system $AX = B$ has a solution X whose entries are symmetric polynomials in $x_1, \dots, x_k$ iff $t+1$ is a 2-power.

A solution of (1.2) or the equation of Theorem 1.3 as a quotient of symmetric polynomials can, of course, be given by Cramer's Rule. The determinants that appear in this quotient are related to Schur polynomials, and so Theorem 1.3 may be interpreted as a result about quotients, mod 2, of Schur polynomials. ([6, p.335])

Definition 1.4. *If $m_1 \ge m_2 \ge \ldots \ge m_k$, the Schur polynomial $s_{m_1,\dots,m_k}(x_1, \dots, x_k)$ is defined by the equation*

$$s_{m_1,\dots,m_k}(x_1,\dots,x_k) \cdot \prod_{1 \le i < j \le k} (x_j - x_i) = \begin{vmatrix} x_1^{m_1+k-1} & x_1^{m_2+k-2} & \cdots & x_1^{m_k+0} \\ & & \vdots & \\ x_k^{m_1+k-1} & x_k^{m_2+k-2} & \cdots & x_k^{m_k+0} \end{vmatrix}.$$

The following corollary is immediate from Theorem 1.3 and Cramer's Rule.

Corollary 1.5. *For $k \ge 3$, let $I = (2^{k-1}-1, \dots, 3, 1)$, $J = (k-1, \dots, 1, 0)$, and for $i \in \{0, \dots, k-1\}$ and $t \ge 2^{k-1}$, let $I_{i,t}$ be obtained from I by removing $2^i - 1$ and placing t at the beginning. Then the quotient $s_{I_{i,t}-J}/s_{I-J}$ of Schur polynomials is a polynomial mod 2 iff $t+1$ is a 2-power.*

We have obtained explicit solutions of (1.2) in several cases. These will be proved in Section 3. The first is the complete solution when $k = 3$.

Theorem 1.6. *Let $m_{i,j,k}$ denote the monomial symmetric polynomial in x_1, x_2, and x_3; i.e., the smallest symmetric polynomial containing $x_1^i x_2^j x_3^k$. The solution of (1.2) when $k = 3$ and $\ell \ge 3$ is given by*

$$\begin{aligned}
p_{0,3,\ell} &= \sum_{\substack{i \ge j \ge k > 0 \\ i+j+k = 2^\ell - 1}} \tbinom{j+k}{k} m_{i,j,k} \\
p_{1,3,\ell} &= \sum_{\substack{i \ge j > 0 \\ i+j = 2^\ell - 2}} (1+j) m_{i,j,0} + \sum_{\substack{i \ge j \ge k > 0 \\ i+j+k = 2^\ell - 2}} \left(1 + \tbinom{j+k}{k-1} + \tbinom{j+k+1}{k+1}\right) m_{i,j,k} \\
p_{2,3,\ell} &= \sum_{\substack{i \ge j \ge k \ge 0 \\ i+j+k = 2^\ell - 4}} \left(1 + \tbinom{j+k+2}{k+1}\right) m_{i,j,k}.
\end{aligned}$$

Incorporating Theorem 1.6 into Theorem 1.1 gives the v_0-, v_1-, and v_2-action, mod higher filtration, in $BP_*(B\mathbb{Z}/2) \otimes_{BP_*} BP_*(B\mathbb{Z}/2) \otimes_{BP_*} BP_*(B\mathbb{Z}/2)$. For example, v_0 acts as

$$(1.7)\quad v_3 m_{4,2,1} + v_4 (m_{12,2,1} + m_{10,4,1} + m_{8,6,1} + m_{9,4,2} + m_{8,5,2} + m_{8,4,3}) + \cdots,$$

where the omitted terms involve v_ℓ for $\ell \ge 5$. The subscripts of the v_3- and v_4-terms for the v_2-action appear in the sentence containing (4.1).

We have also obtained the explicit solution of (1.2) for any k if $\ell = k$.

Theorem 1.8. *The solution of (1.2) when $\ell = k$ has $p_{j,k,k}$ equal to the sum of all monomials of degree $2^k - 2^j$ in $x_1, \dots, x_k$ in which all nonzero exponents are 2-powers. Here $0 \le i \le k-1$.*

This gives the formula for the v_k-component of the BP_*-module structure, modulo higher filtration, of $\bigotimes_{BP_*}^k BP_*(B\mathbb{Z}/2)$. It is complete information, mod higher filtration, for $BP\langle k\rangle$ homology. Johnson-Wilson homology $BP\langle k\rangle$ has coefficients $\mathbb{Z}_{(2)}[v_1,\ldots,v_k]$.([3])

Corollary 1.9. *In $\bigotimes^k_{BP\langle k\rangle_*} BP\langle k\rangle_*(B\mathbb{Z}/2)$, for $0\le j\le k-1$,*

$$v_j\cdot z_I\equiv v_k\sum_E z_{I-E}$$

mod higher filtration, where $E=(e_1,\ldots,e_k)$ ranges over all k-tuples such that all nonzero e_j are 2-powers, and $\sum e_j=2^k-2^j$.

This generalizes [4, Cor 2.7], which says roughly that v_0 acts as $v_k m_{2^{k-1},2^{k-2},\ldots,1}$.

Finally, our most elaborate, and probably most useful, explicit calculation is given in the following result, which gives the complete formula for the v_0-action, mod higher filtration. This is useful since v_0 corresponds to multiplication by 2.

Theorem 1.10. *In $\bigotimes^k_{BP_*} BP_*(B\mathbb{Z}/2)$, v_0 acts as $\sum\limits_{\ell\ge k} v_\ell\cdot p_{0,k,\ell}(x_1,\ldots,x_k)$, where*

$$p_{0,k,\ell}=\sum_f\prod_{i=0}^{\ell-1}x_{f(i)}^{2^i},$$

where f ranges over all surjective functions $\{0,\ldots,\ell-1\}\to\{1,\ldots,k\}$. Equivalently, $p_{0,k,\ell}=\sum m_{|S_1|,\ldots,|S_k|}$, where the sum ranges over all $|S_1|>\cdots>|S_k|$ with $S_1,\ldots,S_k$ a partition of $\{1,2,4,\ldots,2^{\ell-1}\}$ into k nonempty subsets.

See (1.7) for an explicit example when $k=3$.

2. Proof of Theorems 1.1 and 1.3

In this section, we prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. Let $Q=\bigotimes^k_{BP_*} BP_*(B\mathbb{Z}/2)$. Let z_i and z_I be as in the first paragraph of the paper. By [2], Q is spanned by classes $(v_0^{t_0}v_1^{t_1}\cdots)z_I$ with only relations $\sum_{j\ge0}a_jz_{i-j}$ in any factor, where $a_j\in BP_{2j}$ are coefficients in the [2]-series. By [7, 3.17], these satisfy, mod $(v_0,v_1,\ldots)^2$,

$$a_j\equiv\begin{cases}v_i & j=2^i-1,\ i\ge0\\ 0 & j+1\text{ not a 2-power.}\end{cases}$$

Let F_s denote the ideal $(v_0, v_1, \ldots)^s Q$. Then F_s/F_{s+1} is spanned by all $(v_0^{t_0} v_1^{t_1} \cdots) z_I$ with $\sum t_j = s$, with relations

$$\sum_{j \geq 0} v_j z_{i-(2^j-1)} = 0 \tag{2.1}$$

in each factor. As proved in [2, Thm 3.2] (see also [4, 2.3] and our Section 4), this leads to a $\mathbb{Z}/2$-basis for F_s/F_{s+1} consisting of all $(v_k^{t_k} v_{k+1}^{t_{k+1}} \cdots) z_I$ with $\sum t_j = s$.

We claim that if $z_I \in F_0$ and $0 \leq j \leq k-1$, then we must have

$$v_j z_I = \sum_{\ell \geq k} v_\ell p_{j,k,\ell} z_I, \tag{2.2}$$

where $p_{j,k,\ell} = p_{j,k,\ell}(x_1, \ldots, x_k)$ is a symmetric polynomial of degree $2^\ell - 2^j$, acting on z_I by decreasing subscripts as described in the second paragraph of the paper. That the action is symmetric and uniform is due to the uniform nature of the relations (2.1). That it never increases subscripts of z_i is a consequence of naturality: there are inclusions $\bigotimes_{BP_*} BP_*(RP^{2n_i}) \to \bigotimes_{BP_*}^k BP_*(B\mathbb{Z}/2)$ in which the only z_I in the image are those with $i_t \leq n_t$ for all t, and the v_j-actions are compatible.

Note that (2.1) can be interpreted as saying that, for any $i \in \{1, \ldots, k\}$,

$$\sum_{j \geq 0} v_j x_i^{2^j-1} = 0. \tag{2.3}$$

Since the v_ℓ-components are independent if $\ell \geq k$, and (2.2) says that for $j < k \leq \ell$ the v_ℓ-component of the v_j-action is given by the (unknown) polynomial $p_{j,k,\ell}$, we obtain the equation

$$\sum_{j=0}^{k-1} p_{j,k,\ell}(x_1, \ldots, x_k) x_i^{2^j-1} = x_i^{2^\ell-1}$$

for any $i \in \{1, \ldots, k\}$ and $\ell \geq k$. This is the system (1.2). It incorporates all the information of the relations and has a unique solution as a quotient of symmetric polynomials by Cramer's Rule. Our argument shows that it is a polynomial, mod 2.

The v_j-action formula on F_0 applies also on F_s by the nature of the module. ■

Remark 2.4. The proof in [2] that the action of lower v_j's on $\bigotimes_{BP_*}^k BP_*(B\mathbb{Z}/2)$ can be expressed in terms of those with $\ell \geq k$ involved iterative use of the relation (2.1). A precursor of our result here was obtained in [1], motivated by this iteration. In Section 4, we present results of a computer implementation of this iteration, also showing how it fails to yield a polynomial when the exponent on the RHS is not of the form $2^\ell - 1$.

The following lemma, which is central to the proofs of Theorems 1.3 and 1.6, may be of interest in its own right. It involves the complete homogeneous symmetric polynomial of degree d, denoted $h_d(x_1,\dots,x_k)$.

Lemma 2.5. *Let* $f_k(x_1,x_2)=\dfrac{(x_1+x_2)^{k+2}-h_{k+2}(x_1,x_2)}{x_1x_2}$, *a polynomial of degree* k. *Then, mod* 2,

(2.6)

$$h_d(x_1,x_2,x_3)\equiv(x_1+x_2+x_3)\biggl(\sum_{i=0}^{d-1}x_3^i f_{d-1-i}(x_1,x_2)+f_d(x_1,x_2)\sum_{i=0}^{\infty}x_3^{-i-1}(x_1+x_2)^i\biggr).$$

Since $f_d(x_1,x_2)\equiv 0$ *iff* $d+3$ *is a* 2*-power,* $h_d(x_1,x_2,x_3)/(x_1+x_2+x_3)$ *is a polynomial mod* 2 *iff* $d+3$ *is a* 2*-power.*

Proof. One easily verifies that $(x_1+x_2)f_k(x_1,x_2)\equiv f_{k+1}(x_1,x_2)+h_{k+1}(x_1,x_2)$. The RHS of (2.6) expands as

$$\begin{aligned}&\sum_{i=0}^{d-1}x_3^{i+1}f_{d-1-i}(x_1,x_2)+\sum_{i=0}^{d-1}x_3^i(f_{d-i}(x_1,x_2)+h_{d-i}(x_1,x_2))+f_d(x_1,x_2)\\ \equiv\ & x_3^d+\sum_{i=0}^{d-1}x_3^i h_{d-i}(x_1,x_2)=h_d(x_1,x_2,x_3).\end{aligned}$$

■

Proof of Theorem 1.3. By Theorem 1.1, there is a solution if $t=2^\ell-1$. We assume now that $t+1$ is not a 2-power. The augmented matrix of the system in Theorem 1.3 reduces, after several steps, to

$$\left[\begin{array}{cccccc|c}1 & x_1 & x_1^3 & x_1^7 & & x_1^{2^{k-1}-1} & x_1^t\\ 0 & 1 & h_2(x_1,x_2) & h_6(x_1,x_2) & \cdots & h_{2^{k-1}-2}(x_1,x_2) & h_{t-1}(x_1,x_2)\\ 0 & 0 & x_1+x_2+x_3 & h_5(x_1,x_2,x_3) & & h_{2^{k-1}-3}(x_1,x_2,x_3) & h_{t-2}(x_1,x_2,x_3)\\ & & \vdots & & & \vdots & \vdots\\ 0 & 0 & x_1+x_2+x_k & h_5(x_1,x_2,x_k) & \cdots & h_{2^{k-1}-3}(x_1,x_2,x_k) & h_{t-2}(x_1,x_2,x_k)\end{array}\right]$$

By Lemma 2.5, the third row becomes

$$[0\quad 0\quad 1\quad q_4(x_1,x_2,x_3)\quad\cdots\quad q_{2^{k-1}-4}(x_1,x_2,x_3)\mid\phi_{t-3}(x_1,x_2,x_3)],$$

where each q is a polynomial, but ϕ is an infinite series, involving many negative exponents of x_3. It is impossible that there are polynomials $p_0, \ldots, p_{k-1}$ such that

$$p_2 + p_3 q_4 + \cdots + p_{k-1} q_{2^{k-1}-4} = \phi_{k-4}.$$

■

3. Explicit formulas for certain $p_{j,k,\ell}$

In this section, we prove Theorems 1.1, 1.6, and 1.8.

Proof of Theorem 1.6. Similarly to the proof of Theorem 1.3, when $k = 3$, (1.2) is equivalent to

$$\begin{bmatrix} 1 & x_1 & x_1^3 \\ 0 & 1 & h_2(x_1, x_2) \\ 0 & 0 & x_1 + x_2 + x_3 \end{bmatrix} \begin{bmatrix} p_{0,3,\ell} \\ p_{1,3,\ell} \\ p_{2,3,\ell} \end{bmatrix} = \begin{bmatrix} x_1^{2^\ell - 1} \\ h_{2^\ell - 2}(x_1, x_2) \\ h_{2^\ell - 3}(x_1, x_2, x_3) \end{bmatrix}.$$

Thus by Lemma 2.5

$$\begin{aligned} p_{2,3,\ell} &= h_{2^\ell-3}(x_1, x_2, x_3)/(x_1 + x_2 + x_3) \\ &= \sum_{k=0}^{2^\ell - 4} x_3^k \left(\frac{(x_1 + x_2)^{2^\ell - 2 - k} - h_{2^\ell - 2 - k}(x_1, x_2)}{x_1 x_2} \right) \\ &= \sum \left(\tbinom{2^\ell - 2 - k}{j+1} + 1 \right) x_1^{2^\ell - 4 - j - k} x_2^j x_3^k. \end{aligned}$$

Since $\binom{2^\ell-2-k}{j+1} \equiv \binom{j+k+2}{j+1}$, the result for $p_{2,3,\ell}$ follows.

Now we have

$$\begin{aligned} p_{1,3,\ell} &= h_{2^\ell-2}(x_1, x_2) - h_2(x_1, x_2) p_{2,3,\ell} \\ &= \sum x_1^i x_2^{2^\ell - 2 - i} + (x_1^2 + x_1 x_2 + x_2^2) \sum_{\substack{i \geq j \geq k \geq 0 \\ i+j+k = 2^\ell - 4}} (1 + \tbinom{j+k+2}{k+1}) m_{i,j,k}. \end{aligned}$$

If $k > 0$, the coefficient of $m_{i,j,k}$ in this is

$$(1 + \tbinom{j+k+2}{k+1}) + (1 + \tbinom{j+k+1}{k+1}) + (1 + \tbinom{j+k}{k+1}),$$

which equals the claimed value. If $k = 0$ and $j > 0$, there is an extra 1 from the $\sum x_1^i x_2^{2^\ell-2-i}$, and we obtain $\binom{j+2}{1} + \binom{j+1}{1} + \binom{j}{1} \equiv 1 + j$, as desired. The coefficient of $m_{2^\ell-4,0,0}$ is easily seen to be 0.

Finally, we obtain $p_{0,3,\ell}$ from $x_1^{2^\ell-1}+x_1p_{1,3,\ell}+x_1^3p_{2,3,\ell}$. The coefficient of $m_{i,j,0}$ in this is $(1+j)+(1+\binom{j+2}{1})=0$, as desired. If $k>0$, the coefficient of $m_{i,j,k}$ is $(1+\binom{j+k}{k-1}+\binom{j+k+1}{k+1})+(1+\binom{j+k+2}{k+1})\equiv\binom{j+k}{k}$, as desired. ■

Proof of Theorem 1.8. After multiplying the ith row by x_i, we see that it suffices to show that

$$\sum_{i=1}^{k} x_1^{2^{i-1}} g_{2^k-2^{i-1}} = x_1^{2^k}, \tag{3.1}$$

where g_m is the sum of all monomials in $x_1,\dots,x_k$ of degree m with all nonzero exponents 2-powers. (Other rows are handled equivalently.)

The term $x_1^{2^k}$ is obtained once, when $i=k$. The only monomials obtained in the LHS of (3.1) have their x_i-exponent a 2-power for $i>1$, while their x_1-exponent may be a 2-power or the sum of two distinct 2-powers. A term of the first type, $x_1^{2^i}x_2^{2^{t_2}}\cdots x_k^{2^{t_k}}$ with $\sum 2^{t_i}>0$, can be obtained from either the ith term in (3.1) or the $(i+1)$st. So its coefficient is 0 mod 2. A term of the second type, $x_1^{2^a+2^b}x_2^{t_2}\cdots x_k^{t_k}$, can also be obtained in two ways, either from $i=a+1$ or $i=b+1$. ■

Proof of Theorem 1.10. If $i_1,\dots,i_k$ are distinct nonnegative integers, let $V(i_1,\dots,i_k)$ denote the determinant, mod 2, of the Vandermonde matrix

$$\begin{bmatrix} x_1^{i_1} & \cdots & x_1^{i_k} \\ & \vdots & \\ x_k^{i_1} & \cdots & x_k^{i_k}\end{bmatrix}.$$

Note that this equals the mod 2 polynomial $m_{i_1,\dots,i_k}$. By Cramer's Rule applied to (1.2)

$$p_{0,k,\ell}=\frac{V(2,4,8,\dots,2^{k-1},2^\ell)}{V(1,2,4,\dots,2^{k-1})}.$$

The theorem then is a consequence of the following lemma. ■

Lemma 3.2. *For $\ell\ge k$, the only k-tuples $(n_1,\dots,n_k)$ that can be decomposed in an odd number of ways as $n_i=s_i+t_i$ with $(t_1,\dots,t_k)$ a permutation of $(1,2,4,\dots,2^{k-1})$ and $s_i=|S_i|$, where $S_1,\dots,S_k$ is a partition of $\{1,2,4,\dots,2^{\ell-1}\}$ into k nonempty subsets, are the permutations of $(2,4,8,\dots,2^{k-1},2^\ell)$.*

Proof. We will show that all

$$\begin{pmatrix} s_1 & \cdots & s_k \\ t_1 & \cdots & t_k \end{pmatrix} \tag{3.3}$$

as in the lemma can be grouped into pairs with equal column sums $(s_1+t_1,\ldots,s_k+t_k)$ except for permutations (by column) of

$$\begin{pmatrix} 2^0 & 2^1 & \cdots & 2^{k-2} & 2^{k-1}+\cdots+2^{\ell-1} \\ 2^0 & 2^1 & \cdots & 2^{k-2} & 2^{k-1} \end{pmatrix}. \tag{3.4}$$

It is easy to see that (3.4) is the only matrix (3.3) with its column sum. To see how all other matrices occur in pairs, we first consider those of a special type. For the remainder of the proof, we let $e = k-1$, because of the many occurrences of $k-1$.

We say that (3.3) is of Type 1 if, for some i, $t_i = 2^j$ with $j < e$ and $s_i = 2^j + D$ with D the sum of a nonempty set of 2-powers which does not include 2^j. Choose the smallest such j. Then

$$\begin{pmatrix} \cdots & 2^j+D & \cdots & y & \cdots \\ \cdots & 2^j & \cdots & 2^{j+1} & \cdots \end{pmatrix} \text{ is paired with } \begin{pmatrix} \cdots & D & \cdots & 2^j+y & \cdots \\ \cdots & 2^{j+1} & \cdots & 2^j & \cdots \end{pmatrix}.$$

Here the first indicated column in each is column i, while the second indicated column for each is the one in which 2^{j+1} occurs on the bottom row of the first matrix. It is possible that the second indicated column might actually appear to the left of the first one. Other columns are unchanged. Both D and y are nonzero sums of one or more distinct 2-powers not including 2^j. Clearly this method divides into pairs with equal column sums all Type 1 matrices (3.3).

For those (3.3) not of Type 1, we create paths in the matrix (3.3) as follows. We think of entries in the top row as the set of 2-powers comprising its sum. If a top row entry is (the sum of) a set of 2^j's with all $j > e$, we call the smallest such 2^j "unaccompanied," meaning that it is not accompanied by 2-powers of the type which appear on the bottom row. Connect unaccompanied 2-powers to the entry below them. Expand these to the smallest set of paths such that each 2-power in the bottom row for which there are no unconnected 2-powers $\le 2^e$ directly above it is connected to its equal in the top row and to the group of 2-powers $\le 2^e$ directly above it (or to an unaccompanied 2-power $> 2^e$).

Here are two examples. In each case, e is the largest exponent in the bottom row.

Figure 3.5. Flippable example

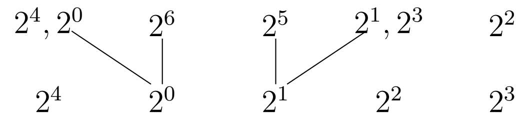
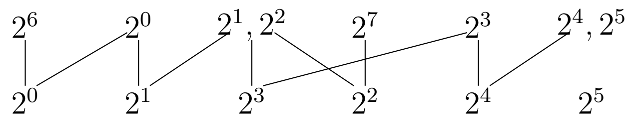


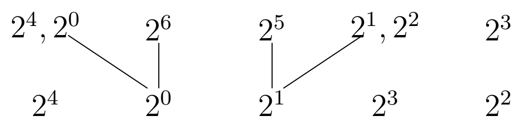
Figure 3.6. Example for reversing 2^e and 2^{e+1}



In Figure 3.6, the bottom 2^3 is considered to be connected to either 2^1 or 2^2 above it.

We begin by illustrating for these two examples the matrix with which they are paired. For Figure 3.5, we can flip vertically the unconnected elements to yield a distinct matrix with the same column sums, as in Figure 3.7.

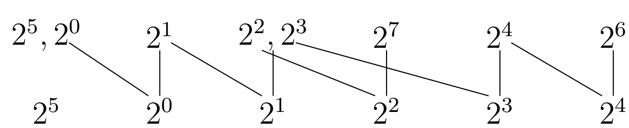
Figure 3.7. Paired with Figure 3.5



Note that if Figure 3.7 were the one being considered, it would lead to Figure 3.5.

For Figure 3.6, only $\binom{2^e}{2^e}$ is unconnected. We follow the path from its accompanying 2^4 until we get to 2^{e+1}, and flip (i.e. move them directly up or down) all the elements $\le 2^e$ along the path, and horizontally reverse the $\binom{2^e}{2^e}$ and $\binom{2^{e+1}}{-}$, obtaining Figure 3.8.

Figure 3.8. Paired with Figure 3.6.



Note that as we follow the path from the 2^4 in the top row of Figure 3.6, when we get to the $2^1, 2^2$ pair, we choose to follow the path through 2^1, so as to get to 2^{e+1} eventually.

Now we consider in general a matrix which is not of Type 1. The end of a path can only be (ignoring 2^j's on top with $j > e$) of one of the two following types.

Figure 3.9. One type of path ending

$$\begin{tikzcd}[column sep=small]
2^a, 2^b \arrow[dr, dash] & \\
2^c & 2^b
\end{tikzcd}$$

Here a, b, and c are distinct 2-powers $\le 2^e$, with possibly more than one 2^a.

Figure 3.10. The other type of path ending

$$\begin{tikzcd}[column sep=small]
2^e, 2^b \arrow[dr, dash] & \\
2^e & 2^b
\end{tikzcd}$$

If there is a path which ends as in Figure 3.9, we can flip all unconnected 2^j's with $j \le e$. Because all connected 2^j's with $j \le e$ are connected (diagonally) top-to-bottom, the unconnected 2^j's with $j \le e$ on top will necessarily be a permutation of those on the bottom, and hence the flipping will not affect the fact that the entries in each row are a permutation of the appropriate set of 2-powers, and it will preserve column sums since flipping is done vertically. Moreover the process is reversible. This case was illustrated in Figures 3.5 and 3.7.

Now we consider the case in which there is no path ending as in Figure 3.9. Then all paths end at the terminus of Figure 3.10. This is exemplified in Figure 3.6, where paths starting at 2^6 and 2^7 both pass through the same position and end up at the same place. If 2^{e+1} is unaccompanied, then vertically flip all the 2^j's in the path from it to the Figure 3.10 terminus, and reverse horizontally the $\binom{2^e}{2^e}$ and $\binom{2^{e+1}}{-}$, as was done in going from Figure 3.6 to 3.8. This will preserve column sums and maintain the permutation property of both rows.

It can happen that 2^{e+1} is accompanied in the middle of a path which ends as in Figure 3.10. An example is:

$$\begin{tikzcd}[column sep=small]
2^6 \arrow[d, dash] & 2^0 \arrow[d, dash] & 2^1, 2^5 \arrow[d, dash] & 2^2 \arrow[d, dash] & 2^3, 2^4 \\
2^0 \arrow[ur, dash] & 2^1 \arrow[ur, dash] & 2^2 \arrow[ur, dash] & 2^3 \arrow[ur, dash] & 2^4
\end{tikzcd}$$

In this case we flip the 2^j's with $j < 2^e$ which occur in the portion of the path between $\binom{2^{e+1}}{-}$ and $\binom{2^e}{2^e}$, and horizontally reverse the $\binom{2^{e+1}}{-}$ and $\binom{2^e}{2^e}$. In this example, it would yield

$$\begin{tikzcd}[column sep=small]
2^6 \arrow[d, dash] & 2^0 \arrow[d, dash] \arrow[dl, dash] & 2^1, 2^4, 2^2 \arrow[dl, dash] \arrow[dr, dash] & 2^3 \arrow[d, dash] \arrow[dr, dash] & 2^5 \arrow[d, dash] \\
2^0 & 2^1 & 2^4 & 2^2 & 2^3
\end{tikzcd}$$

Finally, we consider the case in which 2^{e+1} is accompanied but is not in the middle of a path. Its column cannot contain $\binom{2^j}{2^j}$ with $j < e$, since Type 1 has been ruled out. If its column contains unconnected $\binom{2^a}{2^b}$ with $a \ne b$ and both $\le 2^e$, then this will be part of something flippable, since 2^a and 2^b will have to appear elsewhere, also not in a path. It remains to consider the possibility that the column with 2^{e+1} on top also contains $\binom{2^e}{2^e}$.

If it is of the form

$$\begin{tikzcd}[column sep=small]
2^{e+1}, 2^e, 2^a \arrow[dr, dash] & \\
2^e & 2^a
\end{tikzcd}$$

then there must be a 2^{e+2} elsewhere in the top row (or 2^{e+2} also accompanying 2^{e+1} and 2^{e+3} elsewhere, etc.). Deal with the $\binom{2^{e+2}}{-}$ and $\binom{2^{e+1},2^e}{2^e}$ just as we did with $\binom{2^{e+1}}{-}$ and $\binom{2^e}{2^e}$ earlier.

It cannot happen that $\binom{2^{e+1},2^e}{2^e}$ is accompanied on top by an unconnected 2^a with $a < e$. This would imply an unconnected 2^a elsewhere on the bottom, then an unconnected 2^b above it, etc. However, there must also be an unaccompanied $2^{e+\delta}$ on top somewhere with $\delta \ge 2$, which will lead to a contradiction with the unconnected elements already accounted for.

Finally, there is the possibility that the $\binom{2^{e+1},2^e}{2^e}$ column has nothing else on top except perhaps for some $2^{e+\delta}$ with $\delta \ge 2$. The other columns must have some unconnected flippable parts unless they are all $\binom{2^a}{2^a}$. If there are any $2^{e+\delta}$'s with $\delta \ge 2$, they cannot accompany $\binom{2^a}{2^a}$ with $a < e$ because Type 1 has been ruled out. This leaves as the only possible unpaired matrix the one given in (3.4) or its permutations. (Recall $e = k - 1$.)

■

4. Computer verification

In this section, we describe dramatic computer verifications of how the polynomials of Theorem 1.6 are obtained by iterating relations (2.1), while if the exponent of the analogue of $x^{2^\ell-1}$ is changed to a number not of that form, the iteration doesn't lead to a polynomial.

In [2], [5], and [1], (2.3) was used to replace v_0x_1 or $v_1x_2^1$ or $v_2x_3^3$ by the other terms in the series. This is how they were reduced to elements in $(v_3, v_4, \ldots)Q$. We can implement the replacements simultaneously and iterate in `Maple` and obtain a reduction of an element $v_jz_{(i_1,i_2,i_3)}$, $j \in \{0, 1, 2\}$, to

$$v_3 \sum z_{(i_1-e_1,i_2-e_2,i_3-e_3)} + v_4 \sum z_{(i_1-e_1,i_2-e_2,i_3-e_3)},$$

where the sums are taken over certain triples (e_1, e_2, e_3) summing to $7-(2^j-1)$ (resp. $15-(2^j-1)$). Alternatively we can see what happens if we try a similar algorithm to reduce it to $w \sum z_{(i_1-e_1,i_2-e_2,i_3-e_3)}$ with $\sum e_i = 6-(2^j-1)$. This would correspond to solving (1.2) with $k = 3$ and exponents 6 on the RHS.

For the first, let f_j be a polynomial in z_1, z_2, and z_3 with exponent of z_i corresponding to the subscript in the ith component of z_I. The 5-tuple $(f_0, f_1, f_2, f_3, f_4)$ will represent the coefficients of $v_0, \ldots, v_4$. The quantity $(2^j-1) + \deg(f_j)$ will be constant in a vector and throughout the reduction.

At each step, replace $(f_0, \ldots, f_4)$ by

$$\begin{aligned}&(z_2f_1 + z_3^3f_2,\ z_1^{-1}f_0 + z_3^2f_2,\ z_1^{-3}f_0 + z_2^{-2}f_1,\\ &z_1^{-7}f_0 + z_2^{-6}f_1 + z_3^{-4}f_2 + f_3,\ z_1^{-15}f_0 + z_2^{-14}f_1 + z_3^{-12}f_2 + f_4),\end{aligned}$$

with the convention that a term with negative exponent of any z_i is 0. Note that z_i^{-1} here corresponds to x_i in the first two sections. For example, $(0, f_1, 0, 0, 0)$ is replaced by $(z_2f_1, 0, z_2^{-2}f_1, z_2^{-6}f_1, z_2^{-14}f_1)$, corresponding to the relation $v_1 = x_2^{-1}v_0 + x_2^2v_2 + x_2^6v_3 + x_2^{14}v_4$.

If we start with $(0, 0, z_1^{16}z_2^{16}z_3^{16}, 0, 0)$, during the first 66 iterations there are nonzero entries in at least one of the first three components, but it stabilizes at the 67th iteration to

$$\sum(0, 0, 0, z_1^{16-a_1}z_2^{16-a_2}z_3^{16-a_3}, z_1^{16-b_1}z_2^{16-b_2}z_3^{16-b_3}), \tag{4.1}$$

where (a_1, a_2, a_3) ranges over all permutations of $(4, 0, 0)$, $(2, 2, 0)$, and $(2, 1, 1)$, while (b_1, b_2, b_3) ranges over all permutations of $(12, 0, 0)$, $(10, 2, 0)$, $(8, 4, 0)$, $(6, 6, 0)$, $(10, 1, 1)$, $(9, 2, 1)$, $(6, 5, 1)$, $(8, 2, 2)$, $(6, 4, 2)$, $(5, 5, 2)$, $(6, 3, 3)$, $(5, 4, 3)$, and $(4, 4, 4)$. This is consistent with $p_{2,3,3}$ and $p_{2,3,4}$ in Theorem 1.6, with x_i corresponding to z_i^{-1}. These (b_1, b_2, b_3) are those with $\binom{b_2+b_3+2}{b_3+1} \equiv 0 \bmod 2$.

If we start with a larger monomial in the third component, a similar result is obtained; it just takes a few more iterations. If we start instead with a monomial in the first or second component, the result is similar. The (a_1, a_2, a_3) and (b_1, b_2, b_3) will correspond to the odd binomial coefficients in $p_{0,3,\ell}$ or $p_{1,3,\ell}$ in Theorem 1.6.

Alternatively, omit f_4, and let f_3 correspond to exponent 6 on the RHS of (1.2), rather than 7 or 15. So now (f_0, f_1, f_2, f_3) is replaced by

$$(z_2 f_1 + z_3^3 f_2,\ z_1^{-1} f_0 + z_3^2 f_2,\ z_1^{-3} f_0 + z_2^{-2} f_1,\ z_1^{-6} f_0 + z_2^{-5} f_1 + z_3^{-3} f_2 + f_3).$$

For any initial monomial, the iteration will stabilize to some $(0, 0, 0, q(z_1, z_2, z_3))$ after sufficiently many steps. But the stable vector will involve many terms with exponent of z_3 greater than that of the initial monomial (which our naturality argument precluded when the exponent on the RHS of (1.2) was $2^\ell - 1$), and, as the exponents of the initial vector are increased, the stable vector becomes longer. This yields that the solution of (1.2) when $k = 3$ and the exponent on the RHS is 6 begins as follows, with $m_{i,j}$ denoting the monomial symmetric polynomial in x_1 and x_2.

$$\begin{aligned}
p_0 &= m_{2,1}x_3^3 + m_{4,1}x_3 + (m_{5,1} + m_{3,3}) + (m_{6,1} + m_{5,2} + m_{4,3})x_3^{-1} + m_{7,1}x_3^{-2} \\
&\quad + (m_{8,1} + m_{7,2})x_3^{-3} + \cdots \\
p_1 &= (m_{2,0} + m_{1,1})x_3^3 + (m_{4,0} + m_{2,2})x_3 + (m_{5,0} + m_{3,2}) + (m_{6,0} + m_{5,1} + m_{3,3})x_3^{-1} \\
&\quad + (m_{7,0} + m_{5,2} + m_{4,3})x_3^{-2} + (m_{8,0} + m_{7,1} + m_{6,2})x_3^{-3} + \cdots \\
p_2 &= x_3^3 + (m_{2,0} + m_{1,1})x_3 + m_{2,1} + (m_{4,0} + m_{2,2})x_3^{-1} + (m_{5,0} + m_{4,1} + m_{3,2})x_3^{-2} \\
&\quad + m_{6,0}x_3^{-3} + \cdots .
\end{aligned}$$

The p_2 here gives the first few terms of the second factor on the RHS of (2.6) with $d = 4$.

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