REGULARITY OF VOLUME-MINIMIZING FLOWS ON 3-MANIFOLDS

DAVID L. JOHNSON AND PENELOPE SMITH

Abstract. In [7, 6, 8] the authors characterized the singular set (discontinuities of the graph) of a volume-minimizing rectifiable section of a fiber bundle, showing that, except under certain circumstances, there exists a volume-minimizing rectifiable section with the singular set lying over a codimension-3 set in the base space. In particular, it was shown that for 2-sphere bundles over 3-manifolds, a minimizer exists with a discrete set of singular points.

In this article, we show by analysis of the characterizing horizontal tangent cone, or h-cone, that for a 2-sphere bundle over a compact 3-manifold, such a singular point cannot exist. As a corollary, for any compact 3-manifold, there is a C^1 volume-minimizing one-dimensional foliation. In addition, this same h-cone analysis is used to show that the examples, due to Sharon Pedersen [12], of potentially volume-minimizing rectifiable sections (rectifiable foliations) of the unit tangent bundle to S^{2n+1} are *not*, in fact, volume minimizing.

1. Introduction

In [4], Herman Gluck and Wolfgang Ziller asked which one-dimensional, transversely oriented foliation $\mathcal{F}$ (called a *flow*) on an odd-dimensional round sphere is best-organized, in the sense that the image of the natural section $\xi : M \to T_1(M)$ of the unit tangent bundle, whose value at x is the unit tangent vector of the leaf of $\mathcal{F}$ through x consistent with the orientation of $\mathcal{F}$, has smallest n-dimensional Hausdorff measure.

Their work was in part an effort to interpret the behavior of the Hopf fibration of the three-sphere, and indeed they were able to show that the Hopf fibration did minimize the volume. Specifically, they were able to show that there is a three-form on $T_1(S^3)$ which calibrates the fibers of the Hopf fibrations on S^3, thus those foliations have the least volume of all such flows on the round three-sphere. However, in higher dimensions the Hopf fibrations are not volume-minimizing, and it is likely that volume-minimizing flows on these manifolds are singular. In her thesis [12], Sharon Pedersen illustrated a stable, singular foliation which has much less mass than the Hopf fibration of S^5.

The purpose of the present work is to show that the regularity of Gluck and Ziller's volume-minimizing flow on S^3 is a special case of a theorem that there is a regular (C^1 as a foliation) volume-minimizing flow on compact, oriented 3-manifold. Similarly, there are volume-minimizing sections of the unit tangent bundle (or other $(n-1)$–sphere bundles over n-dimensional manifolds) without isolated poles. As a corollary result, it will follow that Pedersen's currents are *not* volume-minimizing among rectifiable sections of $T_1(S^{2n+1})$.

1.1. Volume of Foliations.

The *volume* of a one-dimensional foliation $\mathcal{F}$ on a compact manifold M can be computed in terms of the Gauss map $\xi : M \to T_1(M)$ defined by mapping x to a unit

Date: May 16, 2011.

2000 *Mathematics Subject Classification.* 49F20, 49F22, 49F10, 58A25, 53C42, 53C65.

Key words and phrases. Geometric measure theory, foliations, sections, volume, minimal submanifolds.

vector $\xi(x)$ tangent to $\mathcal{F}$ at x, which can be chosen consistently if $\mathcal{F}$ is oriented. The formula is given as:

$$\mathcal{V}(\xi) \;=\; \int_M \sqrt{1+\|\nabla\xi\|^2+\cdots+\left\|\nabla\xi^{\wedge(n-1)}\right\|^2}\,dV_M$$

where the vector wedge is interpreted by

$$\nabla\alpha\wedge\nabla\beta(X,Y) := \frac{1}{2}(\nabla_X\alpha\wedge\nabla_Y\beta-\nabla_Y\alpha\wedge\nabla_X\beta),$$

etc., so that

$$(\nabla\xi)^{\wedge k}(X_1,\dots,X_k)=\nabla_{X_1}\xi\wedge\cdots\wedge\nabla_{X_k}\xi.$$

The sum is taken over wedges of order up to $n-1$ since the fiber (S^{n-1}) is $(n-1)$-dimensional. Although this is precisely the n-dimensional Hausdorff measure of the image, which is the mass of the rectifiable current representing the Gauss map as a current in $T_1(M)$, this description has certain advantages.

This definition can be extended to sections σ of any smooth fiber-bundle $B\to M$ with compact fiber F, as defined in [7, 6]. The volume functional is essentially the same, except that the highest-degree term in the square root is the minimum of the dimension of M or that of the fiber,

$$\mathcal{V}(\sigma) \;=\; \int_M \sqrt{1+\|\nabla\sigma\|^2+\cdots+\|\nabla\sigma^{\wedge n}\|^2}\,dV_M,$$

with terms $\left\|\nabla\sigma^i\right\|^2$ being 0 for $i>dim(F)$. The results of this article will apply equally to any S^{n-1}-bundle over a compact, oriented n-manifold M, but the main impetus of the research came out of the original question regarding foliations.

2. Rectifiable Sections.

Let B be a Riemannian fiber bundle with compact fiber F over a Riemannian n-manifold M, with projection $\pi: B\to M$ a Riemannian submersion. F is a j-dimensional compact Riemannian manifold. Following [10], B embeds isometrically in a vector bundle $\pi: E\to M$ of some rank $k\geq j$, which has a smooth inner product $<\,,\,>$ on the fibers, compatible with the Riemannian metric on F. The inner product defines a collection of connections, called *metric connections*, which are compatible with the metric. Let a metric connection ∇ be chosen. The connection ∇ defines a Riemannian metric on the total space E so that the projection $\pi: E\to M$ is a Riemannian submersion and so that the fibers are totally geodesic and isometric with the inner product space $E_x\cong\mathbb{R}^k$ [13], [5].

We will be using multiindices $\alpha=(\alpha_1,\dots,\alpha_{n-l})$, $\alpha_i\in\{1,\dots,n\}$ with $\alpha_1<\cdots<\alpha_{n-l}$, over the local base variables, and $\beta=(\beta_1,\dots,\beta_l)$, $\beta_j\in\{1,\dots,k\}$ with $\beta_1<\cdots<\beta_l$, over the local fiber variables (we will at times need to consider the vector bundle fiber, as well as the compact fiber F; which is considered will be clear by context). The range of pairs (α,β) is over all pairs satisfying $|\beta|+|\alpha|=n$, where $|(\alpha_1,\dots,\alpha_m)|:=m$. As a notational convenience, denote by n the n-tuple $n:=(1,\dots,n)$, and denote the null 0-tuple by 0.

Definition 1. An n-dimensional current T on a Riemannian fiber bundle B over a Riemannian n-manifold M locally, over a coordinate neighborhood Ω on M, decomposes into a collection, called *components*, or *component currents of* T, with respect to the bundle structure. Given local coordinates (x,y) on $\pi^{-1}(\Omega)=\Omega\times\mathbb{R}^k$ and a smooth n-form $\omega\in E^n(\Omega\times\mathbb{R}^k)$, $\omega:=\omega_{\alpha\beta}dx^\alpha\wedge dy^\beta$, define auxiliary currents $E_{\alpha\beta}$ by $E_{\alpha\beta}(\omega):=\int\omega_{\alpha\beta}d\|T\|$, where $\|T\|$ is the measure $\theta\mathcal{H}^n\llcorner Supp(T)$, with $\mathcal{H}^n$ Hausdorff n-dimensional measure in $\Omega\times\mathbb{R}^k$ and θ the multiplicity of T [11, pp 45-46]. The *component currents* of T are defined in terms of *component functions* $t_{\alpha\beta} : \Omega \times \mathbb{R}^k \to \mathbb{R}$ and the auxiliary currents, by:

$$T|_{\pi^{-1}(\Omega)} := \{T_{\alpha\beta}\} := \{t_{\alpha\beta}E_{\alpha\beta}\}\,.$$

The component functions $t_{\alpha\beta} : \pi^{-1}(\Omega) \to \mathbb{R}$ are determined (a.e) by the current T and the pairing between T and an n-form $\omega \in E^n(E)\llcorner\Omega \times \mathbb{R}^k$, given by:

$$T(\omega) := \int_{\Omega\times\mathbb{R}^k} \sum_{\alpha\beta} t_{\alpha\beta}\omega_{\alpha\beta} d\,\|T\|\,.$$

Definition 2. A bounded current T in E is a *(bounded) quasi-section* if, for each coordinate neighborhood $\Omega \subset M$,

(1) $t_{n0} \geq 0$ for $\|T\|$-almost all points $p \in Supp(T)$, that is $< \overrightarrow{T}(q), \mathbf{e}(q) > \geq 0$, $\|T\|$-almost everywhere; where $\mathbf{e}(q)$ is the unique horizontal (that is, perpendicular to the fibers) n-plane at q whose orientation is preserved under π_*.
(2) $\pi_\#(T) = 1[M]$ as an n-dimensional current on M.
(3) $\partial T = 0$ (equivalently, for any $\Omega \subset M$, $\partial\left(T\llcorner\pi^{-1}(\Omega)\right)$ has support contained in $\partial\pi^{-1}(\Omega)$).

Note that each of these conditions is closed under weak convergence. For the first, $t_{n,0} \geq 0$ if $T(\phi) \geq 0$ for all $\phi = \eta dx^1 \wedge \cdots \wedge dx^n$, where η is a smooth, positive function with support in a neighborhood of p. If T_i is a sequence of such currents and $T_i \rightharpoonup T$, then T will also satisfy that condition. Similarly, for the second condition, $\pi_\#(T) = 1[M]$ if and only if $T(\pi^*(dV)) = Vol(M)$, which is again clearly closed under weak convergence. The third condition, likewise, translates as $0 = T(d\phi)$ for all smooth forms ϕ, which is also closed under weak convergence.

Definition 3. There is an $A > 0$ so that the fiber bundle B is contained in the disk bundle $E_A \subset E$ defined by $E_A := \{v \in E\,|\|v\| < A\}$, by compactness of B. Define the space $\widetilde{\Gamma}(E)$ to be the set of all countably rectifiable, integer multiplicity, n-dimensional currents which are quasi-sections in E, with support contained in E_{A}ă, called *(bounded) rectifiable sections* of E, which by the above is a weakly-closed set. The space $\Gamma(E)$ of *(strongly) rectifiable sections of* E is the smallest sequentially weakly-closed space containing the graphs of C^1 sections of E which are supported within E_A.

Thus, a quasi-section which is rectifiable and of integer multiplicity is an element of $\widetilde{\Gamma}(E)$. It would seem to be a strictly stronger condition for it to be in $\Gamma(E)$, however, it is shown in [2] that, over a bounded domain Ω, $\widetilde{\Gamma}(\Omega \times \mathbb{R}^k) = \Gamma(\Omega \times \mathbb{R}^k)$. This extends to the statement that $\widetilde{\Gamma}(E) = \Gamma(E)$ for a vector bundle over a compact manifold M, since any such can be decomposed into finitely many bounded domains where the bundle structure is trivial, by a partition of unity argument.

The space $\widetilde{\Gamma}(B)$ of *rectifiable sections* of B is the subset of $\widetilde{\Gamma}(E)$ of currents with support in B, which is a weakly closed condition with respect to weak convergence. Weak closure follows since, for any point z outside of B, there is a smooth form supported in a compact neighborhood of z disjoint from B. The space $\Gamma(B)$ of *strongly* rectifiable sections is the smallest sequentially, weakly-closed space containing the graphs of C^1 sections of B. Since the fibers of B are compact, as is the base manifold M, minimal-mass elements will exist in $\widetilde{\Gamma}(B)$ or $\Gamma(B)$, and mass-minimizing sequences within any homology class will have convergent subsequences in $\widetilde{\Gamma}(B)$ or $\Gamma(B)$. This follows from lower semi-continuity with respect to convergence of currents, convexity of the mass functional, and the closure and compactness theorems for rectifiable currents. Closure of the conditions of definition (2) under weak convergence will imply that the limits given by the closure and compactness theorems, which are *a priori* rectifiable currents, are indeed rectifiable sections. For compact manifolds, as above, $\widetilde{\Gamma}(E) = \Gamma(E)$, but it is *not* the case that $\widetilde{\Gamma}(B) = \Gamma(B)$ in general (see Proposition (14) below).

Remark 4. A simple modification of the Federer-Flemming closure and compactness theorems shows the following result: [6, 7]

Proposition 5. *Let $\{T_j\} \subset \Gamma(B)$ (resp, $\widetilde{\Gamma}(B)$) be a sequence with equibounded flat norm. Then, there is a subsequence which converges weakly to a current T in $\Gamma(B)$ (resp, $\widetilde{\Gamma}(B)$) .*

Definition 6. Given a current T, the induced measures $\|T\|$ and $\|T_{\alpha\beta}\|$ are defined locally by:

$$\begin{aligned} \|T_{\alpha\beta}\|(A) &:= \sup\left(T_{\alpha\beta}(\omega)\right), \text{ and} \\ \|T\|(A) &:= \sup\left(\sum_{\alpha\beta} T_{\alpha\beta}(\omega)\right), \end{aligned}$$

where the supremum in either case is taken over all n-forms on B, $\omega \in E_0^n(B)$, with $comass(\omega) \leq 1$ [3, 4.1.7] and $Supp(\omega) \subset A$.

2.1. **Crofton's formula.** The usual Crofton's formula (cf. for example [3, 3.2.26]) for the measure of a rectifiable set states that, if W is a rectifiable, Hausdorff n-dimensional set in $\mathbb{R}^{n+k}$, then

$$\mathcal{H}^n(W) = \frac{1}{\beta(n+k,n)} \int_{p\in O^*(n+k,n)} \int_{\mathbb{R}^n} N(p|W,y) d\mathcal{L}^n(y) dV_{O*(N,n)}(p),$$

where $N(p|W,y)$ is the multiplicity at $y \in \mathbb{R}^n$ of the orthogonal projection $p : \mathbb{R}^{n+k} \to \mathbb{R}^n$ restricted to W, $O^*(n+k,n)$ is the space of all such projections with the natural metric of total volume 1, and $\beta(n+k,n) = \int_{p\in O^*(n+k,n)} \|p_*(P)\| \, dV_{O*(n+k,n)}(p)$.

Since the mass of an integer-multiplicity, countably-rectifiable n-current T in $\mathbb{R}^{n+k}$ is the integral with respect to Hausdorff n-dimensional measure restricted to the support of T of the absolute value of the multiplicity θ, the mass of such a T can be represented by essentially the same integral-geometric formula.

Proposition 7. *If T is an integer-multiplicity, countably-rectifiable n-current in $\mathbb{R}^{n+k}$, with multiplicity θ, then the mass of T is given by*

$$\mathcal{M}(T) = \frac{1}{\beta(n+k,n)} \int_{p\in O^*(n+k,n)} \int_{\mathbb{R}^n} N(p|T,y,\theta) d\mathcal{L}^n(y) dV_{O*(n+k,n)}(p),$$

where $N(p|T,y,\theta) = N(p|Supp(T),y)|\theta|$ is the multiplicity at $y \in \mathbb{R}^n$ of the orthogonal projection $p : \mathbb{R}^{n+k} \to \mathbb{R}^n$ restricted to $Supp(T)$, multiplied at each $z \in p^{-1}(y) \cap Supp(T)$ by $|\theta(z)|$, and $O^*(n+k,n)$ is the space of all such projections with the natural metric of total volume 1.

For $T \in \widetilde{\Gamma}(B)$, and $i \in 0,\ldots,n$, set $T_i = \sum_{|\beta|=i} T_{\alpha,\beta}$. T_i is the sum of the components of T that have i vertical directions. Take $x_0 \in M$ and $R > 0$. Set $O^*(E,n,i)$ to be the set of orthogonal projections from $\pi^{-1}(B(x_0,R)) \cong B(x_0,R) \times \mathbb{R}^k \subset \mathbb{R}^{n+k} := E$ which preserve i vertical directions, that is, for which the kernel contains an $\mathbb{R}^{k-i}$ inside of the fiber directions. Any such projection is of course a direct product of projections $p_1 : B(x_0,R) \to \mathbb{R}^{n-i}$ and $p_2 : \mathbb{R}^k \to \mathbb{R}^i$, so

$$O^*(E,n,i) = O^*(\mathbb{R}^n, n-i) \times O^*(\mathbb{R}^k, i).$$

If T is a smooth graph, $T = graph(u)$, then

$$\mathcal{M}(T_i \llcorner \pi^{-1}(B(x_0,R))) = \int_{B(x_0,R)} \left\|\nabla u^{\wedge i}\right\| dV.$$

Proposition 8.

$$\mathcal{M}(T_i \llcorner \pi^{-1}(B(x_0,R))) = \frac{\int_{p\in O^*(E,n,i)} \int_{\mathbb{R}^n} N(p|T,y,\theta) d\mathcal{L}^n(y) dV_{O*(E,n,i)}(p)}{\beta(n,n-i)\beta(k,i)}.$$

Proof. (Compare [11, 3.16])

$$\begin{aligned}
\mathcal{M}(T_i \llcorner \pi^{-1}(B(x_0,R))) &= \sup\{T_i(\phi) \mid comass(\phi)=1\} \\
&= \sup\left\{ T(\phi) \mid comass(\phi)=1, \phi = \sum_{|\beta|=i} \phi_{\alpha\beta} dx^\alpha \wedge dy^\beta \right\} \\
&= \sup\left\{ \int_{Supp(T)} < \overrightarrow{T}(z), \phi > \theta(z) d\|T\| \;\middle|\; \begin{array}{l} comass(\phi)=1, \\ \phi = \sum_{|\beta|=i} \phi_{\alpha\beta} dx^\alpha \wedge dy^\beta \end{array} \right\} \\
&= \frac{\int_{Supp(T)} \int_{p\in O^*(E,n,i)} \left\| p_*(\overrightarrow{T}(z)) \right\| dV_{O*(E,n,i)}(p) \theta(z) d\|T\|}{\beta(n,n-i)\beta(k,i)} \\
&= \frac{\int_{p\in O^*(E,n,i)} \int_{Supp(T)} \left\| p_*(\overrightarrow{T}(z)) \right\| \theta(z) d\|T\| \, dV_{O*(E,n,i)}(p)}{\beta(n,n-i)\beta(k,i)} \\
&= \frac{\int_{p\in O^*(E,n,i)} \int_{\mathbb{R}^n} N(p|T,y,\theta) d\mathcal{L}^n(y) dV_{O*(E,n,i)}(p)}{\beta(n,n-i)\beta(k,i)},
\end{aligned}$$

where the last step follows from the general area-coarea formula. □

3. Existence of horizontal cones

A current $C \in \widetilde{\Gamma}(B(x_0,R) \times F)$ is an *h-cone*, or a *horizontal cone*, at x_0 if $(h_\lambda)_\#(C) = C$. From [7], a *tangent h-cone* at $x_0 \in M$ of a rectifiable section $T \in \widetilde{\Gamma}(B)$ should be the limit of horizontal dilations of T. First, restrict T to $\pi^{-1}(B(x_0,r_0)) \approxeq B(x_0,r_0) \times F$. Then, for $0 < \lambda < r_0$, and $r > 0$, set $h_\lambda : B(x_0,\lambda r) \times F \to B(x_0,r) \times F$ by $h_\lambda(x,v) = (x_0 + (x-x_0)/\lambda, v)$, and set $T_\lambda := (h_\lambda)_\#(T \llcorner B(x_0,\lambda r) \times F)$. In the case where $T = graph(u)$, then T_λ is the graph of u_λ defined by $u_\lambda(x) = u(x_0 + \lambda(x - x_0))$. Then, for a sequence $\lambda_i \downarrow 0$, the h-cone H of T at x_0 is the weak limit $H = \lim_k (h_{\lambda_k})_\# (T \llcorner B(x_0,\lambda_k) \times F)$, if that limit exists. Note that, as $\lambda \downarrow 0$, the curvature of the base will approach 0 and the bundle will become flat. The h-cone is then defined on the Euclidean product $B(x_0,r) \times F \subset \mathbb{R}^n \times F$.

It was shown in [7] that, for mass-minimizing rectifiable sections as constructed in [6], h-cones always exist for some sequence of dilations, since a simple monotonicity result shows that the set of dilations T_λ will have equibounded mass. We provide here a more direct proof of this fact in the case we need. Note that the existence of h-cones is established only for the mass-minimizing currents (with good partial-regularity) shown to exist in [6], which are limits of a sequence of minimizers of functionals with an additional penalty term. It is not known whether other mass-minimizers exist, without the required partial regularity.

For the moment, consider an arbitrary bundle $B \to M$ with compact fiber F. Let T be a "good" mass-minimizing rectifiable section, which is regular over an open dense subset. As before, set $T_i = \sum_{|\beta|=i} T_{\alpha,\beta}$. From [3, 3.3.27],

$$\mathcal{M}(T \llcorner \pi^{-1}(B(x_0,R))) \le \sum_{i=0}^{n} \mathcal{M}(T_i \llcorner \pi^{-1}(B(x_0,R))).$$

This also follows directly from the triangle inequality.

In order to show that a sequence $(h_\lambda)_\#(T \llcorner \pi^{-1}(B(x_0,\lambda R)))$ of stretches converges, we need to show that each component $(h_\lambda)_\#(T_i \llcorner \pi^{-1}(B(x_0,\lambda R)))$ has mass bounded independently of λ.

We use the result from [6, Proposition 4.1], stating that, since T is mass-minimizing and is the limit of penalty-minimizers, every point in *supp*(T) has mass-density at least 1, and satisfies standard monotonicity inequalities, $\mathcal{M}(T \llcorner B(z,\epsilon)) \leq A\epsilon^n$.

Consider $T \llcorner \pi^{-1}(B(x_0,R))$. For each $z \in Supp(T \llcorner \pi^{-1}(B(x_0,R)))$, if $\epsilon > 0$ is sufficiently small, the previous estimate holds on $T \llcorner B(z,\epsilon)$, $\mathcal{M}(T \llcorner B(z,\epsilon)) \leq A_z\epsilon^n$. Since $Supp(T \llcorner \pi^{-1}(B(x_0,\lambda R)))$ is compact, there is a finite subcover $\mathcal{U}$ of such balls, with minimum radius ϵ. Let A be the maximum of the constants A_z for these balls. Now, let $p \in O^*(E,n,i)$. Any ball centered at $y \in Im(p) \subset \mathbb{R}^n$ of radius ϵ will be such that $p^{-1}(y)$ meets finitely many balls in this cover $\mathcal{U}$ (since the whole cover is finite). The mass of the image of each of these balls is less than the mass of the ball in T, since projection is mass-decreasing, so the total image mass within that ball, counting multiplicities, is less than the number of balls in the cover which intersect $p^{-1}(y)$, times $A\epsilon^n$. Thus, there is a constant C so that

$$\mathcal{M}(p_\#(T \llcorner \pi^{-1}(B(x_0,R)))) \leq C\mathcal{L}^n(p(Supp(T \llcorner \pi^{-1}(B(x_0R))))),$$

where $p_\#(T \llcorner \pi^{-1}(B(x_0,R)))$ is the Crofton push-forward current as in §2.1, with multiplicity function $N(p|T,y,\theta)$ at each point in the image.

Similarly,

$$\mathcal{M}(T_i \llcorner \pi^{-1}(B(x_0,R))) \quad \leq \quad C\mathcal{L}^i(p(F))\omega_{n-i}R^{n-i}$$

where $p(F)$ is the image of the fiber F in $\mathbb{R}^i$ (F is a submanifold of $E_x \cong \mathbb{R}^k$), maximized over all $p \in O^*(E,n,i)$. This inequality follows since the image of the projection of T is contained in the image of $F \times B(x_0,R)$.

For precisely the same reasons, with the same constants,

$$\mathcal{M}((h_\lambda)_\#(T_i \llcorner \pi^{-1}(B(x_0,\lambda R)))) \quad \leq \quad C\mathcal{L}^i(p(F))\omega_{n-i}R^{n-i},$$

since the factor of λ coming from the stretch simply expands the image of each projection until it again is contained within the image of $F \times B(x_0,R)$. The conclusion of this argument is the following proposition:

Proposition 9. *$\mathcal{M}((h_\lambda)_\#(T \llcorner \pi^{-1}(B(x_0,\lambda R))))$ is bounded, independently of λ. Thus, given a sequence $\lambda_m \downarrow 0$, a subsequence of $(h_{\lambda_m})_\#(T \llcorner \pi^{-1}(B(x_0,\lambda_m R)))$ converges to a rectifiable section T_0 in $\widetilde{\Gamma}(B(x_0,R) \times F)$.*

Proof. Set $T^{m,R} := (h_{\lambda_m})_\#(T \llcorner \pi^{-1}(B(x_0,\lambda_m R)))$. Then, by taking a diagonal subsequence, for each $j \in \mathbb{Z}$ there is a current $T^j \in \widetilde{\Gamma}(B(x_0,j) \times F)$ so that $(h_{\lambda_m})_\#(T \llcorner \pi^{-1}(B(x_0,\lambda_m j))) \rightharpoonup T^j$ and $T^j \llcorner B(x_0,l) \times F = T^l$, whenever $j > l$, so that there is a current T^0 on $R^n \times F$ which restricts to each of these T^j. □

We now specialize to the case of an S^{n-1}-bundle over a compact n-manifold M.

Proposition 10. *Let $B \to M$ be an $(n-1)$-sphere bundle over a compact n-manifold M. Let T be a good mass-minimizing rectifiable section as before. Assume that $x_0 \in M$ is a pole point of T so that the Hausdorff dimension of the pole is $(n-1)$, that is, that the projection map $\phi_r : S^{(n-1)}(r) \times S^{n-1} \to S^{n-1}$, inducing a Crofton projection $(\phi_r)_\#(T \llcorner S^{(n-1)}(r) \times S^{n-1}) \in \mathbf{R}^{n-1}(S^{n-1})$, has limit having positive $(n-1)$-dimensional mass A for some subsequence of the sequence $r_m = \lambda_m R$. Then the current T^0 of Proposition (9) will be an h-cone.*

Proof. Since each T^j minimizes the scaled and stretched functional

$$\mathcal{V}^j(S) := \mathcal{V}((h_{\lambda_j}^{-1})_\#(S))/\left(\lambda_j R\mathcal{M}((\phi_{\lambda_j R})_\#(T\llcorner S^{n-1}(\lambda_j R)\times S^{n-1})\right),$$

T^0 will minimize the limiting functional

$$\begin{aligned}\mathcal{V}_0(S\llcorner B^n(x_0,R)\times S^{n-1}) &= \lim \mathcal{V}^j(S\llcorner B^n(x_0,R)\times S^{n-1})\\ &= \mathcal{M}(S_{n-1}\llcorner B^n(x_0,R)\times S^{n-1}),\end{aligned}$$

where $S_{n-1} := \sum_{|\alpha|=1} S_{\alpha\beta}$ is that part of the current S which has $(n-1)$ vertical components, one horizontal component. The stretched functionals $\mathcal{V}^j$, as $j\to\infty$, magnify the terms with more vertical components by the effect of $(h_{\lambda_j}^{-1})_\#$, and under the assumption that the pole at x_0 has Hausdorff dimension $(n-1)$ that highest-order term will dominate all others in the normalized limit. This reduces to

$$\int_{B(x_0,R)} \left\|\nabla u^{\wedge(n-1)}\right\| dV,$$

if S is a smooth graph $S = \mathit{graph}(u)$. Note also that $\mathcal{M}^{n-1}(S_{n-1}\llcorner\partial B^n(x_0,R)\times S^{n-1})$ is the $(n-1)$-dimensional mass of the projection $(\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})$. Since T^0 minimizes, for any R

$$\begin{aligned}\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1}) &\le \mathcal{V}_0(C(T^0\llcorner\partial B^n(x_0,R)\times S^{n-1}))\\ &= R\mathcal{M}^{n-1}(T^0_{n-1}\llcorner\partial B^n(x_0,R)\times S^{n-1})\\ &= R\mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right),\end{aligned}$$

where $C()$ denotes the h-cone over the boundary $T_0\llcorner\partial B^n(x_0,R)\times S^{n-1})$. On the other hand, by slicing

$$\begin{aligned}\frac{d}{dR}\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1}) &\ge \mathcal{M}^{n-1}(T^0_{n-1}\llcorner\partial B^n(x_0,R)\times S^{n-1})\\ &= \mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right),\end{aligned}$$

so that

$$\begin{aligned}&\frac{d}{dR}\frac{\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1})}{R}\\ =\ &\frac{\frac{d}{dR}\left(\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1})\right)R - \mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1})}{R}\\ \ge\ &\frac{\mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right)R - \mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1})}{R}\\ \ge\ &0,\end{aligned}$$

and so $\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1})/R$ is an increasing function of R. However, since T^0 is invariant at least under the sequence of stretches by h_{λ_j}, the projected mass $\mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right)$ must be the same for $R=\lambda_j R_0$, so that the values of $\mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right)$ repeat over the intervals $[\lambda_{j+1}R_0, \lambda_j R_0]$ and the increasing function $\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1})/R$ satisfies

$$\frac{\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1})}{R} \le \mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right)$$

so

$$\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1})/R \le \inf\left(\mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right)\right).$$

However, since

$$\begin{aligned}\mathcal{V}_0(T^0 \llcorner B^n(x_0,R)\times S^{n-1}) &\geq \int_0^R \mathcal{M}^{n-1}\left((\phi_r)_\#(T^0\llcorner S^{n-1}(r)\times S^{n-1})\right) dr \\ &\geq R\inf\left(\mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right)\right),\end{aligned}$$

all of these inequalities must be equalities, and necessarily $\mathcal{M}^{n-1}\left((\phi_R)_\#(T^0\llcorner S^{n-1}(R)\times S^{n-1})\right)$ must be constant. Moreover,

$$\mathcal{V}_0(T^0\llcorner B^n(x_0,R)\times S^{n-1}) = \mathcal{V}_0(C(T^0\llcorner \partial B^n(x_0,R)\times S^{n-1})),$$

and since any change with respect to the radial direction (of positive measure) would introduce a strict inequality in that integral, $T^0\llcorner B^n(x_0,R)\times S^{n-1} = C(T^0\llcorner\partial B^n(x_0,R)\times S^{n-1})$ T^0-almost everywhere. Thus T^0 is an h-cone. □

The *degree* of a rectifiable section $C\in\widetilde{\Gamma}(S^{n-1}\times S^{n-1})$ is defined by

$$deg(C) := \int_{I(C)} dV_{S^{n-1}} = \int_C \Pi_2^*(dV_{S^{n-1}}) = C(\Pi_2^*(dV_{S^{n-1}})),$$

which is clearly a weakly closed condition. If C is the graph of a smooth map $\phi: S^{n-1}\to S^{n-1}$, then $deg(C)=deg(\phi)$, and in particular, if ϕ is the restriction of a smooth map $\Phi: B^n\to S^{n-1}$ to the boundary, then $deg(C)=0$. By taking transfinite limits, if C arises from the h-cone of a strongly rectifiable section $S\in\Gamma(B^n\times S^{n-1})$, $deg(C)=0$ since C is a weak limit of degree-zero currents.

Definition 11. The *degree* of a pole point $x_0\in M^n$ of a rectifiable section $S\in\Gamma(B)$, where B is an S^{n-1}-bundle over M, $deg(S,x_0)$, is the degree of the restriction of an h-cone ψ of S to the boundary $\psi\llcorner S^{n-1}(r)\times S^{n-1}$.

Theorem 12. *If B is an $(n-1)$-sphere bundle over a compact n-manifold M, and if $T\in\widetilde{\Gamma}(B)$ is a smooth graph except on a finite set of fibers $\pi^{-1}(x_i)$, so that the degree of each singular fiber is* 0 *and so that there is an h-cone at each fiber, then $T\in\Gamma(B)$.*

Proof. The only part of this statement requiring proof is that, in a neighborhood of each singular fiber, the current is a limit of smooth currents. Certainly, if the degree of any of the singular fibers is nonzero it cannot be in $\Gamma(B)$. If the degree is 0, however, since the graph is smooth within the boundary spheres $S^{n-1}(r)\times S^{n-1}$ of $B\llcorner\pi^{-1}(B(x_0,r))\cong B(x_0,r)\times S^{n-1}$, the h-cone is a cone over a current $S\in\Gamma(S^{n-1}(1)\times S^{n-1})$, in fact, S is the limit of the smooth sequence of stretches of $T\llcorner S^{n-1}(r)\times S^{n-1}$. Since the degree of the singularity is 0, each graph $T\llcorner S^{n-1}(r)\times S^{n-1}$ is (smoothly) homotopic to the constant map, mapping $S^{n-1}(r)$ to $p_0\in S^{n-1}$. If $H(x,t): S^{n-1}(r)\times[0,1]\to S^{n-1}$ is that homotopy, then the graph $G(y): B(x_0,r)\to S^{n-1}$ defined by $G(y)=H(ry/|y|, 1-|y|/r)$ will be a smoothable graph which can be extended to a section of B agreeing with T outside of this neighborhood. Clearly, given a sequence $r_i\to 0$, the maps

$$T_i = \begin{cases} G_{r_i}, & d(x_0,x)<r_i \\ T, & d(x_0,x)\geq r_i\end{cases}$$

will be a sequence of currents converging weakly to T, which are smooth in a neighborhood of the pole point x_0. Since there are finitely many singular points of T by hypothesis, iterating this construction will generate a sequence of smooth currents converging to T. □

4. Rectifiable foliations, rectifiable sections.

Consider now the case where B is the subbundle $T_1(M)$ of unit vectors in $T_*(M)$. The connection used to define the metric on $T_*(M)$ restricts to an associated connection on $T_1(M)$, since the connection is a metric connection, and defines a metric on $T_1(M)$ as before.

Rectifiable 1-dimensional foliations on M are rectifiable sections of $T_*(M)$ whose support lies within $T_1(M)$. As above, this condition will be weakly closed, so that the Federer closure and compactness theorems hold.

Theorem 13. [7, 6] *For any homology class of sections in $\widetilde{\Gamma}(T_1(M))$, there is a mass-minimizing rectifiable foliation $\mathcal{F}$ with support which is the Gauss map of a C^1 graph over an open, dense subset of M.*

The *regular points* of a rectifiable foliation S correspond to points where the Gauss map is continuous, and singularities, or *pole points*, are points $x \in M$ where the Gauss map is discontinuous. Equivalently, pole points are those $x \in M$ for which the set $Supp(S) \cap \pi^{-1}(x)$ consists of more than one point. Points of $Supp(S)$ lying over pole points are called *pole elements*.

4.1. **The degree of a pole point.** Let S be a mass-minimizing rectifiable section of $T_1(M^3)$. By [8], there is such a minimizer with only a finite number of pole points, each of which contains the entire fiber in the support of S. Note that, in[8], the results need to be modified to indicate that [6] does not show that any minimizer has the required smoothness, only that there is one minimizer with the claimed partial regularity. Assume that S is such a minimizer. The question of regularity of a mass-minimizing section of $T_1(M^3)$ becomes whether such a pole point can exist.

If x_0 is an isolated pole point of $S \in \widetilde{\Gamma}(T_1(M^n))$, by Proposition (10) there is an h-cone centered at x_0. By Theorem (12), $S \in \Gamma(T_1(M^n))$. In addition, the h-cone at x_0 is a rectifiable section $\psi \in \Gamma(B^n \times S^{n-1})$, when restricted to a ball of radius 1 in the base. Slicing the h-cone ψ by a cylinder of radius r generates a rectifiable current C in $S^{n-1}(r) \times S^{n-1} \cong S^{n-1} \times S^{n-1}$ for almost any r by slicing theory. Since ψ is an h-cone, however, C is independent of r, thus the slice is rectifiable for all r, and so is in $\Gamma(S^{n-1} \times S^{n-1})$ as a bundle over the first factor. The key to existence of such a singularity is the degree of the current C.

Since S has no interior boundary, and by [8] the support of S contains all of $\pi^{-1}(x_0)$, the *image* $I(C)$ of C, defined as the push-forward image $(\Pi_2)_\#(C)$, for $\Pi_2 : S^{n-1} \times S^{n-1} \to S^{n-1}$ the projection onto the second factor (the fiber), must have support the entire sphere.

We now return to the claim in Section 2 that not all weak rectifiable sections are strong rectifiable sections.

Proposition 14. $\widetilde{\Gamma}(T_1(S^2)) \neq \Gamma(T_1(S^2))$.

Proof. Since there are no continuous sections of $T_1(S^2)$, that is, $\Gamma(T_1(S^2)) = \emptyset$, it suffices to show that $\widetilde{\Gamma}(T_1(S^2)) \neq \emptyset$. Given a point $p \in S^2$, and $v \in T_1(S^2, p)$, translate v parallel to itself along longitudes to $-p$. The rectifiable section generated by this procedure will have a singular point at $-p$, with the entire fiber of the sphere bundle in the support over $-p$. Since there is no boundary and it projects to $1[S^2]$ on $S^2 \backslash \{-p\}$, it is an element of $\widetilde{\Gamma}(T_1(S^2))$. □

Remark 15. Of course, this current is an element of $\Gamma(T_*(S^2))$, and is the limit of a sequence of smooth vector fields, each of which has a zero of degree 2 at $-p$, with length 1 outside of neighborhoods of $-p$. It should also be noted that this topological obstruction is not the only way that it can be possible for $\widetilde{\Gamma}(B) \neq \Gamma(B)$ for B an $(n-1)$-sphere bundle on an n-manifold. Since the degree of an isolated singularity is local, it follows that any isolated singularities of $T \in \Gamma(B)$ will have degree 0. But even on a sphere bundle B with global smooth sections, it is easy to construct singular sections with two isolated singularities, one of degree 2 and the other of degree -2. Such singular sections are clearly in $\widetilde{\Gamma}(B)\backslash\Gamma(B)$.

5. Non-Existence of Isolated Singularities

Now that we have shown that an isolated pole of a volume-minimizing section S of $T_1(M)$ necessarily stretches to an energy-minimizing section S_0 for the limiting volume $\mathcal{V}_0$, we proceed to show that it cannot exist if the degree of the pole is 0. As before, set $S_0 \in \widetilde{\Gamma}_R(B(0,r) \times S^{n-1})$ to be an h-cone of S, and set $C := S_0 \llcorner S^{n-1} \times S^{n-1}$ (r may be assumed to be larger than 1).

Theorem 16. *If S is a volume-minimizing rectifiable section of $T_1(M)$ which is continuous over an open, dense subset of M, then S cannot have a degree-zero isolated pole point x_0, with supp$(S) \cap \pi^{-1}(x_0) = S^{n-1} = \pi^{-1}(x_0)$.*

Proof. Let S be a mass-minimizing section which is continuous over an open, dense subset, as guaranteed by [6], as discussed above. Assume that S has a degree-zero isolated singularity x_0, with the entire fiber contained in the support of S. There is a h-cone S_0 of S at x_0 by §3. The current $C = S_0 \llcorner S^{n-1}(r) \times S^{n-1}$ has degree 0, as in §4.1, and so there is a rectifiable current F so that $\partial F = C - graph(constant)$ in $S^{n-1} \times S^{n-1}$. In fact, the h-cone S_0 can be used to construct such a current F_0 which is a "rectifiable homotopy", that is, which extends to a rectifiable section on $\left(S^{n-1} \times I\right) \times S^{n-1}$as an n-dimensional current with $\partial F = C \times 0 - S^{2n-1} \times \{pt\} \times 1$. For each i in a sequence $S_i \in \widetilde{\Gamma}(B(x_0,1) \times S^{n-1})$ converging to the h-cone S_0, and for each r, $S_i \llcorner \partial B(x_0,r) \times S^{n-1} = S_i(r) \llcorner S^{n-1} \times S$ is a smooth graph of degree 0, so there is a rectifiable current "fence" $F_i(r)$ of dimension n so that $\partial F_i(r) = S_i(r) \llcorner S^{n-1} \times S - graph(constant) \llcorner S^{n-1} \times S$. Since $S_i \rightharpoonup S_0$, which is a cone, $F_i(r)$ can be chosen with bounded mass, so there is a convergent subsequence with limit $F_0(r)$. Since S_0 is an h-cone, it may be assumed that $F_0(r) = (h_r)_\#(F_0(1))$. The current

$$S_r := S_0 \llcorner B(x_0,R)\backslash B(x_0,r) \times S^{n-1} + F_0(r) + graph(constant) \llcorner B(x_0,r)$$

has the same boundary as $S_0 \llcorner B(x_0,R)$. However, $S_0 \llcorner B(x_0,R)$ minimizes the limiting functional $\mathcal{V}_0$, so, independent of r, $\mathcal{V}_0(S_r) \geq \mathcal{V}_0(S_0)$. But,

$$\begin{aligned} \mathcal{V}_0(S_r) &= \mathcal{V}_0\left(S_0 \llcorner B(x_0,R)\backslash B(x_0,r) \times S^{n-1} + F_0(r) + graph(constant) \llcorner B(x_0,r)\right) \\ &= \mathcal{V}_0\left(S_0 \llcorner B(x_0,R)\backslash B(x_0,r) \times S^{n-1}\right) + \mathcal{V}_0\left(F_0(r)\right), \end{aligned}$$

since $\mathcal{V}_0(graph(constant)) = 0$. In addition, $\mathcal{V}_0\left(F_0(r)\right) = A$ is independent of radius since $\mathcal{V}_0(F_0(r))$ is the mass of the image of $F_0(r)$ under the projection onto S^{n-1}. However,

$$\mathcal{V}_0(S_0 \llcorner B(x_0,R)) - \mathcal{V}_0\left(S_0 \llcorner B(x_0,R)\backslash B(x_0,r) \times S^{n-1}\right) = Br$$

since S_0 is an h-cone. The constants A and B do not depend upon R, except for the limitation that $r < R$. Clearly, for R sufficiently large $\mathcal{V}_0(S_0 \llcorner B(x_0,R)) - \mathcal{V}_0(S_r) = Br - A$ will eventually be positive for some r large enough, contradicting the fact that $S_0 \llcorner B(x_0,R)$ minimizes $\mathcal{V}_0$ there. □

Corollary 17. *If M is a compact 3-manifold, then there is a volume-minimizing one-dimensional foliation of class C^1.*

Proof. By [8], there is a volume-minimizing rectifiable section of $T_1(M)$ with only isolated singular points, for which the support of each contains the entire fiber. Such isolated poles cannot exist by the theorem, so there is a rectifiable section with no poles, so that the section is continuous on all of M. Since that section is the tangent field of the foliation, the foliation is of class C^1. □

Remark 18. Sharon Pedersen, in [12], defined, for each $n \geq 1$, a rectifiable section P_n of $T_1(S^{2n+1})$, defined by parallel translation of a unit vector $v \in T_1(S^{2n+1}, x)$ along meridians to $-x$. This is a rectifiable foliation and a minimal submanifold except over a single point, and was shown to have, for $n > 1$, much smaller volume than the foliations defined by the standard Hopf fibrations. She conjectured that this current might minimize volume amongst rectifiable sections of $T_1(S^{2n+1})$, but this is not the case as shown below.

Corollary 19. *The rectifiable sections P_n of $T_1(S^{2n+1})$ are not volume-minimizing rectifiable foliations.*

Proof. The singularity at $-x$ of such a foliation is precisely the kind shown to not exist by Theorem (16). □

References

[1] F. J. Almgren, Jr. *Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two* , Bull. (New Series) Am. Math. Soc., **8** (1983), 327–328.

[2] A. Coventry, The graphs of smooth functions are dense in the space of Cartesian currents on a smooth bounded domain, but not every Cartesian current is the limit of such graphs, Research reports (Mathematics), number CMA-MRR 45- 98, Australian National University Publications, 1998.

[3] Herbert Federer, *Geometric Measure Theory,* Springer-Verlag 1969.

[4] Herman Gluck and Wolfgang Ziller, *On the volume of a unit vector field on the three-sphere,* Commentarii Mathematici Helvitici, **61** (1986), 177–192.

[5] D. L. Johnson, *Kähler submersions and holomorphic connections,* Jour. Diff. Geo. **15** (1980)ă, 71–79.

[6] D. L. Johnson and P. Smith, *Partial regularity of mass-minimizing rectifiable sections,* Annals of Global Analysis and Geometry, **30** (2006), 239-287.

[7] D. L. Johnson and P. Smith, *Regularity of volume-minimizing graphs,* Indiana University Mathematics Journal, **44** (1995), 45–85.

[8] D. L. Johnson and P. Smith, *Regularity of mass-minimizing one-dimensional foliations,* Analysis and Geometry on Foliated Manifolds, Proceedings of the VII International Colloquium on Differential Geometry, (1994), World Scientific, 81–98.

[9] Gary Lawler, *A sufficient condition for a cone to be area-minimizing,* Memoirs of the American Mathematical Society **91** (1991).

[10] J. D. Moore and R. Schlafly, *On equivariant isometric embeddings,* Mathematische Zeitschrift **173** (1980), 119–133.

[11] F. Morgan, *Geometric Measure Theory, A Beginner's Guide*, Academic Press, 1988; second edition, 1995.

[12] Sharon Pedersen *Volumes of vector fields on spheres,* **336** (1993) Trans. Am. Math. Soc., 69–78.

[13] Takeshi Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds* **10** (1958), Tôhoku Math. J., 338–354.

Current address: Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015-3174

E-mail address: `dlj0@lehigh.edu, ps02@lehigh.edu`