SECTIONAL CURVATURE AND CURVATURE NORMAL FORMS

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INTRODUCTION

The classical notion of the sectional curvature $r(P)$ of a plane section $P \subseteq T_*(M, m)$ of a Riemannian manifold M can be defined as the Gaussian curvature at m of the surface $\mathrm{Exp}_m(P)$ of M obtained by spraying out along all geodesics tangent to P. This gives r: $G(2, T_*(M, m)) \to \mathbf{R}$, the sectional curvature function on the Grassmann manifold of 2-planes in $T_*(M,m)$, which determines to what extent the local geometry of M differs from that of Euclidean space. The Riemannian curvature tensor R is then defined by polarization of the quadratic r. It is this tensor, rather than the more geometric sectional curvature, that has had the broader application. For example, many strong theorems have been developed relating R to global topological characteristics of M, in particular the characteristic algebra and the Betti numbers. As r determines R, in theory at least there is a strong relationship between properties of r and these topological characteristics. However, this relationship is obscured as properties of r do not translate easily into properties of R.

The purpose of this paper is to relate the behavior of r to that of R. The approach taken is inspired by the similar situation for a symmetric operator T on $\mathbf{R}^n$, where the eigenvectors and eigenvalues correspond to critical points and values of the function $t(x) = \langle Tx, x\rangle$ on the unit sphere. For the Riemannian curvature tensor, abstracted to a symmetric operator on $\Lambda^2(V)$, the sectional curvature r replaces the function t as the critical points and values of r have clear geometric meaning. The primary question is, to what extent do these "characteristic" points and values of r determine R?

Definition: Let $\mathscr{R}(n)$ denote the space of all algebraic curvature tensors on an n-dimensional real inner product space V, and let $\mathscr{P}$ be a subset of $\mathscr{R}(n)$. A curvature tensor $R \in \mathscr{P}$ satisfying the first Bianchi identity *has a normal form relative to* $\mathscr{P}$ if there exists a set $\{P_i\}$ of critical points of the sectional curvature function $r_R : G(2,V) \to \mathbf{R}$ such that; if $R' \in \mathscr{P}$ satisfies the first Bianchi identity, has $\{P_i\}$ as critical points of the sectional curvature r_R, and has $r_{R'}(P_i) = r_R(P_i)$ for all i, then $R' = R$. The collection $\{(P_i, A_i)\}$, where $A_i = r_R(P_i)$, is then called a *normal form* for R. Explicit reference to $\mathscr{P}$ will be omitted where there is no ambiguity.

Remark. The assumption that R satisfies the first Bianchi identity is essential, as r does not even determine R algebraically without it.

Unfortunately, not all curvature tensors possess normal forms relative to $\mathscr{R}(n)$ in this sense. For that reason, this paper will primarily consider Kähler curvature

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tensors, where the situation becomes more tractable due to the somewhat simplified algebra.

In [17] Singer and Thorpe show that any Einstein operator in dimension 4 has a normal form relative to all curvature tensors in that dimension. Their result depends on the existence of a large number of critical points, in a special array, which follows from the specific algebraic situation. In the case of Kähler operators more standard techniques will guarantee the existence of sufficient numbers of critical points. In particular, a generic set of all Kähler curvature tensors have sectional curvature functions that are essentially Morse functions, which allows determination of lower bounds on the number of distinct critical points [Theorem (3.2)]. Algebraic and topological arguments are then applied to yield a normal form theorem for Kähler operators in dimensions 4 and 6 under the assumption of positive sectional curvature [Theorem (4.2)].

In [17], Singer and Thorpe conjecture that, if R is in an irreducible invariant subspace of the space $\mathscr{R}(n)$ of all curvature tensors in dimension n, then R has a normal form relative to that subspace, since operators in irreducible subspaces ought to be of a simpler type. They establish this for $n = 4$ and for all but the space of Weyl tensors in higher dimensions.

A similar decomposition holds for Kähler operators [18]; results analogous to Singer and Thorp's are easily obtained in the Kähler case. In addition, many nonalgebraic properties of the manifold M are reflected in the normal form of R. As an example, the normal form of a Kähler homogeneous space of low dimension is explicitly computed.

Remark. Using methods developed in this research, though not specifically the normal form theorems, the author has recently been able to verify the Hopf conjecture (nonnegative curvature implies nonnegative Euler characteristic) for six-dimensional Kähler manifolds [8].

The results obtained here comprise the major portion of the author's doctoral thesis at M.I.T. I would like to express my thanks to my advisor, I. M. Singer, who guided the work on this paper. I also wish to thank M.I.T. for the support it provided.

1. ALGEBRAIC PRELIMINARIES

Definition 1.1. An *algebraic curvature tensor* R on a real inner product space V of dimension n is a symmetric linear operator R on $\Lambda^2(V)$, where $\Lambda^2(V)$ is given the naturally-induced inner product. The space of all such operators is denoted by $\mathscr{R}(n)$.

$\mathscr{R}(n)$ is naturally an inner product space with $\langle R, S\rangle = \operatorname{trace}(RS)$. The induced inner product on $\Lambda^2(V)$ is given by $\langle v \wedge w, x \wedge y\rangle = \langle v,x\rangle\langle w,y\rangle - \langle v,y\rangle\langle w,x\rangle$.

Definition 1.2. $R \in \mathscr{R}(n)$ is *proper* if R satisfies the first Bianchi identity. That is, given an orthonormal basis $\{v_i\}$ of V, for

$$R_{ijk\ell} = \langle R(v_i \wedge v_j), v_k \wedge v_\ell\rangle, \quad R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0.$$

The Grassmann manifold $G(2,n)$ of oriented 2-planes in V can be identified with the space of unit-length decomposable bivectors $v \wedge w \in \Lambda^2(V)$ in an obvious fashion. The *sectional curvature function* r_R: $G(2,n) \to \mathbf{R}$ is then defined as $r_R(P) = \langle RP,P\rangle$. Reference to R will be omitted where no ambiguity should result.

Remark. It goes without saying that these notions correspond with their geometric antecedents if $V = T_*(M,m)$.

Consider now the analogous situation for Kähler manifolds. Let V be a hermitian complex vector space, of real dimension $2n$, with complex structure automorphism J: $V \to V$. J may be considered as an element of $\mathscr{R}(2n)$ by $J(v \wedge w) = Jv \wedge Jw$ as the inner product is hermitian.

Definition 1.3. $R \in \mathscr{R}(2n)$ is called a *Kähler curvature tensor* if $JR = RJ = R$. The space of all Kähler curvature operators in dimension $2n$ forms a linear subspace $\mathscr{K}(n)$ of $\mathscr{R}(2n)$. $\mathscr{K}(n)$ is then naturally an inner product space by restriction of the inner product on $\mathscr{R}(2n)$.

Remark. The major reason, in the context of this paper, for primarily considering Kähler manifolds is that the condition simplifies the behavior of r somewhat.

Definition 1.4. An oriented 2-plane $P \in G(2,2n)$ is called *holomorphic* if $J(P) = P$. $\{P \mid P \text{ is holomorphic}\} \simeq \mathbf{C}P^{n-1} \times \mathbf{Z}_2 = \pm\mathbf{C}P^{n-1}$. $P \in \overset{+}{(-)}\,\mathbf{C}P^{n-1}$ accordingly as $P = \overset{+}{(-)}\, v \wedge Jv$, $v \in P$ any unit vector. The *holomorphic sectional curvature* of $R \in \mathscr{K}(n)$ is $r_{R|\pm\mathbf{C}P^{n-1}}$. P is called *antiholomorphic* if $\langle JP,P\rangle = 0$.

Recalling that $\Lambda^2(V) \simeq o(2n)$ under the usual identification $v \wedge w \to T_{v\wedge w}$, where $\langle T_{v\wedge w}x,y\rangle = \langle v \wedge w, x \wedge y\rangle$, $u(n) \subset o(2n)$ is identified with

$$\{\xi \in \Lambda^2(V) \mid J\xi = \xi\}.$$

Note the sign convention here differs from that of [9]. Definition (1.3) then states that $R \in \mathscr{K}(n)$ if and only if $R(o(n)) \subset u(n)$. If $\{v_1, Jv_1 = v_{1*}, v_2, \ldots, v_{n*}\} = \{v_\alpha\}$ is any hermitian orthonormal basis of V (such a basis will be called *unitary*), the element $I \in \Lambda^2(V)$ corresponding to $J \in o\ (2n)$ is given by $I = \sum_{i=1} v_i \wedge v_{i*}$. One easily sees that $\langle P,JP\rangle = \langle P,I\rangle^2$, thus $\langle P,I\rangle = 1$ if and only if P is holomorphic, and $\langle P,I\rangle = 0$ if and only if P is antiholomorphic.

For $R \in \mathscr{R}(2n)$, and $x \wedge y \in \Lambda^2(V)$, define $R_{xy} \in o(2n) \simeq \Lambda^2(V)$ by $\langle R_{xy}z,w\rangle = \langle R(x \wedge y), z \wedge w\rangle$. Then, following [17], define b: $\mathscr{K}(n) \to \mathscr{R}(2n)$ by $b(R)_{xy}z = R_{xy}z + R_{yz}x + R_{zx}y$; ρ: $\mathscr{K}(n) \to \mathrm{Sym}(V)$, the space of symmetric operators on V, by

$$\langle \rho(R)v,w\rangle = \frac{1}{2}\,[\mathrm{trace}\,(u \to R_{vu}w) - \frac{1}{2}\,\mathrm{trace}\,(R(v \wedge Jw)\circ J)],$$

and $\quad$ tr: $\mathscr{K}(n) \to \mathbf{R}$ by $\mathrm{tr}\,(R) = \dfrac{1}{2}\,\mathrm{trace}\,(\rho(R))$.

An easy calculation verifies that, for proper operators, the definitions of ρ and tr agree with those of [17].

$U(n)$ acts on $\Lambda^2(V)$, hence on $\mathscr{K}(n)$, by restriction of the action of $O(2n)$. b, ρ, and tr above are all easily seen to be equivariant, thus the following subspaces are $U(n)$-invariant:

$$\mathscr{S} = \ker(b)^{\perp}\,;\, \mathscr{I} = \ker(\mathrm{tr})^{\perp}\,; \quad \mathscr{W} = \ker(\rho) \cap \ker(\mathrm{tr});$$

and

$$\mathscr{B} = \ker(\rho)^{\perp} \cap \ker(\mathrm{tr}).$$

As in [17] or [10], it can be shown that these subspaces are irreducible and mutually orthogonal, with $\mathscr{K}(n) \simeq \mathscr{S} \oplus \mathscr{I} \oplus \mathscr{W} \oplus \mathscr{B}$. $\mathscr{I}$ consists of multiples of the operator $R_0 \in \mathscr{K}(n)$ corresponding to the curvature tensor of $\mathbf{C}P^n$ with the Fubini-Study metric. That is, if $\langle R(v_\alpha \wedge v_\beta),\, v_\gamma \wedge v_\delta \rangle = R_{\alpha\beta\gamma\delta}$ for α, β, γ, $\delta \in \{1, 1^*, \ldots, n^*\}$, $R_{0ii*ii*} = 1$, $R_{0ii*jj*} = 1/2$, $R_{0ijij} = R_{0ij*ij*} = 1/4$, and all other terms zero except those determined by these values and the symmetries of $R_{\alpha\beta\gamma\delta}$. The space $\mathscr{W}$ is the space of *Bochner-Weyl* tensors, and R is *Einstein* if $R \in \mathscr{I} \oplus \mathscr{W} = \mathscr{E}$, which is equivalent to the Ricci tensor $\rho(R) = \lambda(Id)$. The following proposition is an easy calculation along the lines of [17] and [18].

PROPOSITION 1.5.

(i) *$R \in \mathscr{I} \oplus \mathscr{W} \oplus \mathscr{B} = \mathscr{S}$ if and only if R is proper.*

(ii) *$R \in \mathscr{S}$ if and only if the holomorphic sectional curvature vanishes.*

(iii) *$R \in \mathscr{W}$ if and only if $\rho(R) = 0$ and R is proper, if and only if R is proper and $R(I) = 0$.*

(iv) *$R \in \mathscr{I} \oplus \mathscr{W}$ if and only if R is proper and $R(I) = \lambda I$.*

(v) *$R \in \mathscr{W} \oplus \mathscr{B}$ if and only if R is proper and $\langle R(I), I \rangle = 0$.*

The following proposition can be found in [13].

PROPOSITION 1.6. *Define.* σ: Sym$(V) \to \mathscr{K}(n)$ *by*

$$\sigma(T)(v \wedge w) = (1/2(n+2))\,[Tv \wedge w + v \wedge Tw + JTv \wedge Jw + Jv \wedge JTw - 2 < Tv, Jw > J \\ + 2 < Jv, w > JT + -(\mathrm{tr}(T)/2(n+1))(v \wedge w + Jv \wedge Jw - 2 < v, Jw > J)].$$

Then

$$\sigma\colon \mathrm{Sym}(V) \to \mathscr{I} \oplus \mathscr{B} \text{ and } \sigma = (\rho_{|\mathscr{I} \oplus \mathscr{B}})^{-1}$$

Note. J and JT are viewed in context as either operators on V or elements of $\Lambda^2(V)$. Also, as above, the signs differ from [13] due to the conventions used.

2. GENERAL FORM OF A CRITICAL POINT

The purpose of this section is to determine algebraic conditions on $R(P)$, $R \in \mathscr{R}(n)$, equivalent to P being a critical point of r_R. In Section 4 these conditions will be exploited to develop normal form theorems.

In Section 1 the Grassmannian $G = G(2,n)$ was considered as a submanifold of $\Lambda^2(V)$. Previous attempts to analyze R in terms of the critical behavior of r_R viewed G in this context and applied Lagrange multipliers [17], [19]. However, in dimensions greater than 4 the codimension of G in $\Lambda^2(V)$ is too large, and the algebraic conditions become cumbersome. However, G sits naturally as a complex hypersurface of $\mathbf{C}P^{n-1}$ [9, pp. 278–282]. In this context, using Lagrange multipliers, the proper conditions fall out easily.

The embedding $\phi\colon G \to \mathbf{C}P^{n-1}$ is defined as follows: for $P = v \wedge w$, $\{v,w\}$ an oriented orthonormal basis of P, $\phi(P) = [v + iw]$, where $v + iw \in V \otimes_{\mathbf{R}} \mathbf{C}$ and [ ] denotes the residue in $\mathbf{C}P^{n-1}$. The image of ϕ is evidently the variety $\sum_{\alpha=1}^{n} (z_\alpha)^2 = 0$ where, for an orthonormal basis $\{v_\alpha\}$ of V, $z \in V \times \mathbf{C}$ is given by $z_\alpha v_\alpha = z$.

Note: The usual Einstein summation conventions (with all indices lowered) will hold throughout. For the bigraded indices $\alpha\beta$ in $\Lambda^2(V)$ the sum will be taken over all $\alpha < \beta$.

r may now be described in terms of the homogeneous coordinates of $\mathbf{C}P^{n-1}$ restricted to G. If $P = v \wedge w = (x_\alpha v_\alpha) \wedge (y_\beta v_\beta) = a_{\alpha\beta} v_\alpha \wedge v_\beta$, for any $z \in V \otimes \mathbf{C}$ such that $[z] = \phi(P)$, $z = (a_\alpha + ib_\alpha) v_\alpha$, $a_{\alpha\beta} = 2(a_\alpha b_\beta - a_\beta b_\alpha)/(z \cdot \bar{z})$. Thus

$$r(P) = a_{\alpha\beta} a_{\gamma\delta} R_{\alpha\beta\gamma\delta} = 4(a_\alpha b_\beta - a_\beta b_\alpha)(a_\gamma b_\delta - a_\delta b_\gamma) R_{\alpha\beta\gamma\delta}/(z \cdot \bar{z})^2.$$

Extend r to all of $\mathbf{C}P^{n-1}$ via the same formula, then lift to a function on $\mathbf{C}^n - \{0\}$, denoting the lifting by $\hat{r}$. $W = \pi^{-1}(G)$ is the smooth affine variety $\sum_\alpha z_\alpha^2 = 0$, where π is the canonical projection.

PROPOSITION 2.1. *$r_{|G}$ has a critical point at $P \in G$ if and only if $d\hat{r}(z) = 0$ for any $z \in \pi^{-1}([z])$.*

Proof. $r_{|G}$ has $[z]$ as a critical point if and only if $\hat{r}_{|W}$ has z as a critical point, for any lift z. By Lagrange multipliers, z is critical if and only if

$$\begin{aligned} d\hat{r}(z) &= c_1(a_\alpha da_\alpha - b_\alpha db_\alpha) + c_2(b_\alpha da_\alpha + a_\alpha db_\alpha) \\ &= c_1(\bar{a}, -\bar{b}) + c_2(\bar{b}, \bar{a}). \end{aligned}$$

Let $g(z) = 4(a_\alpha b_\beta - a_\beta b_\alpha)(a_\gamma b_\delta - a_\delta b_\gamma) R_{\alpha\beta\gamma\delta}$. Using the standard induced inner product on $V \otimes \mathbf{C}$, for $(\bar{a}, 0) = a_\alpha da_\alpha$, etc., $\langle dg(z), (\bar{a},0)\rangle = 2g(z)$ and $\langle dg(z), (\bar{b},0)\rangle = 0$. Now, at a critical point of $\hat{r}_{|W}$, $\langle d\hat{r}(z), (\bar{a},0)\rangle = c_1|\bar{a}|^2$; however,

$$\langle dr(z), (\bar{a},0)\rangle = \langle dg(z)/(z\cdot\bar{z})^2, \quad (\bar{a},0)\rangle - 4\hat{r}(z) < (\bar{a},\bar{b}), (\bar{a},0)\rangle / z\cdot\bar{z} = 0;$$

thus $c_1 = 0$. Similarly, $c_2 = 0$.

Now, to obtain an explicit algebraic condition for P to be a critical point of r, for $z \in \pi^{-1}(P)$ choose $z = (a_\alpha + ib_\alpha) v_\alpha$, where $v = a_\alpha v_\alpha$ and $w = b_\alpha v_\alpha$, as above. Then the equation $d\hat{r} = 0$ becomes

PROPOSITION 2.2. *A plane $P = v \wedge w = (a_\alpha v_\alpha) \wedge (b_\beta v_\beta) = a_{\alpha\beta} v_\alpha \wedge v_\beta$ is* *a critical point of r_R with critical value $A = r(P)$ if and only if, for each $t \in \{1, \ldots, n\}$,*

$$a_t A = \sum_{\beta \neq t, \gamma < \delta} b_\beta a_{\gamma\delta} R_{t\beta\gamma\delta}$$

$$b_t A = \sum_{\alpha \neq t, \gamma < \delta} a_\alpha a_{\gamma\delta} R_{\alpha t\gamma\delta}.$$

If $R(P)$ is considered as a skew-symmetric operator on V, then by extension to an operator on $V \otimes \mathbf{C}$, $R(P)$ is skew-hermitian. From this point of view, Proposition (2.2) becomes

COROLLARY 2.3. *r_R has a critical plane P with critical value A if and only if, for P considered as an (equivalence class of) element(s) of $V \otimes \mathbf{C}$,*

$$R(P)P = -iAP.$$

Using Lagrange multipliers in $\Lambda^2(V)$, J. Thorpe has shown that, for $R \in \mathscr{R}(n)$, $P \in G(2,n)$ is a critical point of r_R if and only if $R(P) = AP + S(P)$ for some $S \in \mathscr{S}$ [19]. In [17] this is specialized to $RP = AP + B * P$ as the Hodge star spans $\mathscr{S}$. $* \Lambda^k(V) \to \Lambda^{n-k}(V)$ is defined in general, given a unit vector $\xi \in \Lambda^n(V)$, by $\langle *\nu, \mu\rangle = \langle \xi, \nu \wedge \mu\rangle$. When V is complex, ξ will always be chosen to be $v_1 \wedge v_{1*} \wedge \ldots \wedge v_n \wedge v_{n*}$.

In the remainder of this section the special properties of the Kähler case will be applied to yield more information on $R(P)$ when P is a critical point of r_R. For $n = 2$, 3 results analogous to Singer and Thorpe's are obtained. The first result is well-known [1].

PROPOSITION 2.4. *If $R \in \mathscr{K}(n)$, any critical plane of $r_{R|\mathbf{C}P^n}$ is a critical plane of r_R.*

Not all critical planes are holomorphic, in fact the more interesting critical planes are nonholomorphic. For a plane $P \in G - \pm \mathbf{C}P^{n-1}$, choose a unitary basis $\{v_\alpha\}$ of V so that $P = av_1 \wedge v_{1*} + bv_1 \wedge v_2$. $b \neq 0$ as P is nonholomorphic. If $\xi = (P + JP)/|P + JP|$, then $\xi \in u(n)$, $|\xi| = 1$, and, as an operator on V, $\operatorname{rank}_{\mathbf{R}}(\xi) = 4$ and $\det_{\mathbf{C}}(\xi_{|(\ker\xi)^\perp}) > 0$. Furthermore, any such ξ is evidently of the form $(P + JP)/|P + JP|$, for some $P \in G - \pm\mathbf{C}P^{n-1}$, as there is a unitary basis of V so that $\xi = av_1 \wedge v_{1*} + bv_2 \wedge v_{2*}$ with $a > 0$, $b < 0$ and $a^2 + b^2 = 1$. Set $P = \dfrac{1}{\sqrt{1-2ab}}(\sqrt{a}\,v_1 + \sqrt{-b}v_{2*}) \wedge (\sqrt{a}\,v_{1*} + \sqrt{-b}v_2)$.

The following proposition is straightforward.

PROPOSITION 2.5. *Define η: $(G - \pm \mathbf{C}P^{n-1}) \to \psi$,*

$$\psi = \{\xi \in u(n) | \operatorname{rank}_{\mathbf{R}}(\xi) = 4, \quad \det_{\mathbf{C}}(\xi_{|(\ker\xi)^\perp}) > 0, \textit{ and } |\xi| = 1\}$$

by $\eta(P) = (P + JP)/|P + JP|$. Then η is a submersion, and, for $P = v \wedge w$,

$$\eta^{-1}(\eta(P)) = \{(\cos tv + \sin tJv) \wedge (\cos tw + \sin tJw)\} \mid t \in [0, 2\pi]\} \simeq S^1.$$

The Kähler identities imply that, if $Q \in \eta^{-1}(\eta(P))$, then $r_R(Q) = r_R(P)$ for $R \in \mathscr{K}(n)$. Thus $r_{R|G-\pm\mathbf{C}P^{n-1}}$ projects to $\bar{r}_R: \psi \to \mathbf{R}$. As r and η are both real-analytic, so is $\bar{r}$. Furthermore, if P is a critical plane of r_R, so is any $Q \in \eta^{-1}(\eta(P))$, and P is a critical plane if and only if $\eta(P)$ is a critical point of $\bar{r}_R$. Thus r_R is not a Morse function if there are any nonholomorphic critical planes. The proper analogue of nondegeneracy for these functions will be developed in the next section.

Definition 2.6. Two nonholomorphic planes P, Q will be called *distinct* if $\eta(P) \neq \pm \eta(Q)$. Two holomorphic planes are *distinct* if $P \neq \pm Q$.

In low dimensions a critical plane of a Kähler curvature function r_R is very nearly an eigenvector of R, in fact, if $R \in \mathscr{K}(2)$ and Q is a nonholomorphic critical zero, then $Q \in \ker(R)$. This also holds for $R \in \mathscr{W} \subset \mathscr{K}(3)$. Of primary use here and in [8] is the following

THEOREM 2.7. *Let $R \in \mathscr{K}(n)$ be proper, let P be a holomorphic critical plane of r_R with critical value A, and let Q be a nonholomorphic critical plane with critical value B. Then*:

(1) *If $n = 2$, $R(P) = AP + A' * P$, where $A + A' = \langle RP, I\rangle$.*

$$RQ = B(Q + JQ - \langle Q, I\rangle \mathrm{I}), \text{ and } \langle RQ, I\rangle = 0.$$

(2) *If $n = 3$, there are holomorphic planes P', P'' such that P, P', P'' are mutually orthogonal and $R(P) = AP + A'P' + A''P''$, with $A + A' + A'' = \langle R(P), I\rangle$. $R(Q) = B(Q + JQ - \langle Q,I\rangle I_Q) + \langle RQ,I\rangle(- * Q \wedge JQ/|Q \wedge JQ|)$, where*

$$I_Q = I + * Q \wedge JQ/|Q \wedge JQ|.$$

Note. For $n = 3$, $- * Q \wedge JQ/|Q \wedge JQ| \in + \mathbf{C}P^2$. If $Q = av_1 \wedge v_{1*} + bv_1 \wedge v_2$ (as above), $- * Q \wedge JQ/|Q \wedge JQ| = v_3 \wedge v_{3*}$.

Proof. The first sentence of part (1) follows from [17] and the definition of I. Proposition (2.2) yields the second sentence in (1) after a short computation. The first sentence in part (2) is obtained by diagonalizing the skew-hermitian operator $R(P)$. Corollary (2.3) implies that P is an eigenvector; the equation follows. The remainder follows by part (1) and Proposition (2.2).

COROLLARY 2.8. *If $R \in \mathscr{E} \subseteq \mathscr{K}(n)$ with $R(I) = \lambda I$, and if*

(1) *$n = 2$, then $R(P) = AP + (\lambda - A) * P$. Also, if $\lambda \neq 0$, Q must be antiholomorphic with $Q + JQ$ an eigenvector of R, having eigenvalue $2B$.*

(2) *$n = 3$, then*

$$RQ = B(Q + JQ - \langle Q,I\rangle I_Q) - \lambda \langle Q,I\rangle(* Q \wedge JQ/|Q \wedge JQ|).$$

3. GENERIC NUMBERS OF CRITICAL POINTS FOR KÄHLER CURVATURE FUNCTIONS

In order to establish a normal form theorem for Kähler curvature tensors, it is necessary: (1) to establish lower bounds on the number of distinct critical points of the sectional curvature r; and (2) to assure that the conditions on R given by these various critical points determine R, which corresponds to the critical points being in a general position. The first of these problems is amenable to standard Morse-theoretic techniques, and will be resolved in this section. The second is a more specialized construction, and will be handled in Section 4. Both problems require generiticity assumptions, hence the normal form results obtained will only hold for a generic class of such operators.

It has been noted that, for $R \in \mathscr{K}(n)$, r_R is not generally a Morse function. There is, however, a natural analogue of nondegeneracy for this class of functions.

Definition 3.1. For $R \in \mathscr{K}(n)$, r_R will be called *nondegenerate* if all holomorphic critical points of r_R and all critical points of $\bar{r}_R$: $\psi \to \mathbf{R}$ are nondegenerate. For brevity, R itself will also be called nondegenerate in this case.

Remark. It is evident that, if r is nondegenerate all critical circles $\eta^{-1}(\eta(Q))$, for Q a nonholomorphic critical plane, will be nondegenerate critical manifolds [4].

THEOREM 3.2. *The set of all $R \in \mathscr{K}(n)$ such that r_R is nondegenerate contains an open dense subset of $\mathscr{K}(n)$.*

Proof. It suffices to show that, for a given holomorphic plane P (respectively, a given nonholomorphic plane Q), there is an $R' \in \mathscr{K}(n)$ with P (resp., Q) as a nondegenerate critical point. For then, as $r_{R+R'} = r_R + r_{R'}$, and by the analyticity of $\det((r_{R+\varepsilon R'})_{**}) = \det[r_{R**} + \varepsilon r_{R'**}]$, if R has P (resp., Q) as a degenerate critical point, $r_{R+\varepsilon R'}$ will have nonsingular hessian for almost all ε near 0. Thus R may be arbitrarily closely approximated by an operator with P (resp., Q) as a nondegenerate critical point. By recursion, R can be approximated by a nondegenerate R''.

Let $P = v_1 \wedge v_{1*}$, by an appropriate choice of unitary basis. Then, if R is given by $R_{11*11*} = 1$, all other terms 0, r_R has P as a nondegenerate critical point. Similarly, if $Q = av_1 \wedge v_{1*} + bv_1 \wedge v_2$ with $a \neq 0$, choose R by $R_{11*11*} = 0$, $R_{11*12} = a/b$, $R_{1212} = 1 - (a/b)^2$, $R_{12*12*} = (a/b)^2$, $R_{11*22*} = 1$, and all other terms 0 except those that must be nonzero to satisfy the Kähler conditions. $\bar{r}_R$ will have $\eta(Q)$ as a nondegenerate critical point unless $a = 0$, in which case R can be chosen by $R_{1212} = 1$, $R_{12*12*} = -1$, all other terms zero.

Remark. Similarly, the critical values may be assumed to be distinct. Also, note that a similar statement may be obtained for $\mathscr{S}^{\perp} = \{R \in \mathscr{K}(n) \mid R$ is proper$\}$, as the examples are all proper. Similarly $\{R \in \mathscr{K}(n) \mid r_{|\mathbf{C}P^{n-1}}$ is a Morse function$\}$ contains an open dense subset of $\mathscr{K}(n)$.

Using Theorem (3.2) lower bounds for the number of distinct critical points of r may now easily be computed, for r nondegenerate. Consider r as a function on the unoriented Grassmannian $\bar{G} = G(2,2n)/P \simeq -P$. As $r(-P) = r(P)$ this is well-defined. $P \in G$ will be considered as an element of $\bar{G}$ as well as G in the obvious way. Note that r is nondegenerate as a function on $\bar{G}$ if and only if it is as a function on G.

Let r be nondegenerate, and let Q be a nonholomorphic critical point of r. $C = \eta^{-1}(\eta(Q))$ is the critical circle of r containing Q. Let

$$J_Q = \{P \in \bar{G} \mid r(P) \leq r(Q)\}, J_Q^- = \{P \in \bar{G} \mid r(P) < r(Q)\}.$$

By [4, Theorem 1], the critical groups with coefficient ring $\mathbf{Z}_2$ ($\mathbf{Z}_2$ coefficients are used throughout this section) are:

$$H_*(J_Q \cap V, J_Q^- \cap V) \simeq H_{*-\lambda}(C) = \begin{cases} \mathbf{Z}_2, & *-\lambda=0,\ 1 \\ 0, & \text{else} \end{cases}$$

where V is a suitable neighborhood of C in $\bar{G}$, and λ is the index of the hessian of $\bar{r}$ at Q.

For a holomorphic critical plane P, let J_p, J_p^- be defined as above. Then

$$H_*(J_p \cap V, J_p^- \cap V) \simeq \begin{cases} \mathbf{Z}_2, & * = \lambda \\ 0, & \text{else} \end{cases}$$

if λ is the Morse index of r at P. By [16, Theorem 11.1],

$$\sum_P \operatorname{rank}(H_*(J_p \cap V, J_p^- \cap V)) \geq \sum_k \operatorname{rank}(H_k(\bar{G}, \mathbf{Z}_2)),$$

where the sum is taken over all distinct critical points of r. The homology of $\bar{G}$ is well-known; in the cases under consideration in this paper,

$$\text{for } n = 2, \quad \sum_k \operatorname{rank}(H_k(\bar{G}(2,4), \mathbf{Z}_2)) = 6,$$

$$\text{for } n = 3, \quad \sum_k \operatorname{rank}(H_k(\bar{G}(2,6), \mathbf{Z}_2) = 15$$

Now if $n = 2$, $r_{|\mathbf{C}P^1}$ must have at least 2 critical points. If $n = 3$, $r_{|\mathbf{C}P^2}$ must have at least 3. This establishes the following

PROPOSITION 3.3. *If $R \in \mathscr{K}(n)$ is nondegenerate, then*:

(i) *for $n = 2$, r_R must have at least 4 distinct critical planes, at least 2 of which must be holomorphic,*

(ii) *for $n = 3$, r_R must have at least 9 distinct critical planes, at least 3 of which must be holomorphic.*

Remark. Without additional assumptions it is difficult to find lower bounds on the number of distinct nonholomorphic critical planes, or indeed to ascertain that any must exist. This can, however, be resolved for $n = 2$.

PROPOSITION 3.4. *If $R \in \mathscr{K}(2)$ is proper, and if* $\operatorname{tr}(R) \neq 0$, *then, for $\alpha = \{Q \in G(2,4) - (\pm \mathbf{C}P^1) \mid \langle Q, R(I)\rangle = 0\}$, Q is a nonholomorphic critical plane of r_R if and only if $Q \in \alpha$ and Q is a critical point of $r_{R|\alpha}$.*

Proof. As in Theorem (2.7), $\langle R(Q), I\rangle = 0$ if Q is critical, thus any such critical points are in α. Let Q be given by $Q = av_1 \wedge v_1 + bv_1 \wedge v_2$, as usual. If Q

is a critical point of $r_{|\mathfrak{a}}$, then $R(Q) = AQ + B * Q + T$, for appropriate A and B, where T is tangent to G at Q. T can then be written as

$$T = cv_1 \wedge v_{2*} + Dv_2 \wedge v_{1*} + E(av_1 \wedge v_2 - bv_1 \wedge v_{1*}) + F(av_{1*} \wedge v_2 + bv_2 \wedge v_{2*})$$

[17]. As $R(Q) \in u(2)$ and $\langle R(Q), I\rangle = 0$, $C = D$, $A = -B$, and $E = F$. $\pi(R(I))$ spans the normal space to $\mathfrak{a}$ at Q, where π is the projection onto

$$T_*(G(2,4), Q) = \{\xi \in \Lambda^2(V) \mid \langle Q, \xi\rangle = 0, \xi \wedge Q = 0\}.$$

As Q is a critical point of $r_{|\mathfrak{a}}$, $\pi(R(I)) = \lambda T$, which, with the above expression for T, implies $T = 0$ as $\mathrm{tr}(R) \neq 0$.

Remark. From this a standard limiting argument will show that any proper $R \in \mathscr{K}(2)$ has at least 2 distinct nonholomorphic critical planes.

In this case where r_R is positive (resp., negative) stronger results may be obtained on the number of distinct nonholomorphic critical planes. Note that, in the Kähler case the assumption that $r_R > 0$ is a nontrivial assumption on the behavior of r, as a general $R \in \mathscr{K}(n)$ cannot be altered to one with positive curvature by adding an operator with constant curvature.

Actually, the necessary condition is somewhat weaker than strictly positive curvature. Let $\mathscr{K}(n)^+ = \{R \in \mathscr{K}(3), R \text{ proper} \mid \rho(R_{|W}) \text{ is positive-definite, for all complex 2-dimensional subspaces } W \subseteq V\}$. ($\mathscr{K}(n)^-$ is defined analogously.) Note that $R \in \mathscr{K}(2)^+$ if and only if $\rho(R)$ is positive-definite and R is proper.

Remark. The results obtained below relative to $\mathscr{K}(n)^+$ may easily be modified to apply to $\mathscr{K}(n)^-$ as well.

PROPOSITION 3.5. *If $R \in \mathscr{K}(2)^+$, then r_R has at least 3 distinct nonholomorphic critical planes, Q_i. Moreover, $\eta(Q_i)$ are mutually orthogonal, if R is nondegenerate.*

Proof. Let $\eta(\mathfrak{a}) = \{\eta(Q) \mid Q \in \mathfrak{a}\}$. The conditions of the Proposition imply that $\eta(\mathfrak{a}) \simeq S^2$, as $\eta(\mathfrak{a}) = \{\xi \in u(2) \mid \langle \xi, R(I)\rangle = 0, |\xi| = 1\}$, since $\rho(R)$ is positive definite, so $\langle R(I), \xi\rangle = 0$ implies that $\det_{\mathbf{C}}(\xi) > 0$. $\bar{r}_{|\eta(\mathfrak{a})}$, being a quadratic on the subspace generated by $\eta(\mathfrak{a})$, must have at least 3 critical points. That $\eta(Q_i)$ are orthonormal follows from Theorem (2.7).

In higher dimensions the picture is somewhat more complicated. Consider now $n = 3$. For any nonholomorphic plane Q, the 4-vector $Q \wedge JQ$ may be viewed as a complex 2-dimensional subspace of V. Let $G(2, Q \wedge JQ) \subseteq G(2,2n)$ be the space of planes in $Q \wedge JQ$. Let $\mathfrak{A} \subset G(2,6)$ be defined by

$$\mathfrak{A} = \{Q \in G(2,6) \mid \langle Q, R(I_Q)\rangle = 0\}.$$

If $R \in \mathscr{K}(3)^+$, $\mathfrak{A} \subset G - (\pm \mathbf{C}P^2)$.

PROPOSITION 3.6. *Let $R \in \mathscr{K}(3)^+$. Then $\mathfrak{A}$ is a compact, real-analytic locally-trivial fibration over $\mathbf{C}P^2$, where the projection π: $\mathfrak{A} \to \mathbf{C}P^2$ is defined by $\pi(Q) = - * Q \wedge JQ/|Q \wedge JQ|$.*

Remark. The fiber $\mathfrak{A}_Q = \{P \in G(2, Q \wedge JQ) \mid \langle RP, I_Q\rangle = 0\}$ is clearly diffeomorphic to $\mathfrak{a}$ above, for $R \in \mathscr{K}(2)^+$.

Proof. As f: $G(2,6) - (\pm \mathbf{C}P^2) \to \mathbf{R}$ defined by $f(Q) = \langle Q, R(I) \rangle$. Satisfies $df \neq 0$ on $f^{-1}(0) = \mathfrak{A}$, $\mathfrak{A}$ is analytic. It is evident that π is a submersion by a direct calculation; the Proposition is easily completed by standard arguments.

Consider $\eta(\mathfrak{A})$. $\pi \circ \eta^{-1} : \eta(\mathfrak{A}) \to \mathbf{C}P^2$ is well-defined as $\eta^{-1}(\eta(Q)) \subset \pi^{-1}(\pi(Q))$. This is also a compact, analytic, locally-trivial fibration, with fiber diffeomorphic to $\eta(\alpha) \simeq S^2$. By an argument paralleling that of Theorem (3.2),

$$\{R \in \mathscr{K}(3)^+ \mid r_R \text{ is nondegenerate}\}$$

contains an open dense subset of $\mathscr{K}(3)^+$.

PROPOSITION 3.7. *Let $R \in \mathscr{K}(3)^+$. Then, any critical point of $r_{R|\mathfrak{A}}$ is a nonholomorphic critical plane of r_R, and any nonholomorphic critical plane of r_R is on $\mathfrak{A}$.*

Proof. Theorem (2.7) yields the last clause. Now, let Q be a critical point of $r_{|\mathfrak{A}}$. Then Q is a critical point of $r_{|\mathfrak{A}_Q}$, so Q is a critical point of $r_{|G(2,Q\wedge JQ)}$, applying Proposition (3.4). $T_*(G(2,6),Q) = T_*(G(2,Q \wedge TQ),Q) + T_*(\mathfrak{A},Q)$ (sum not direct) as the normal space to $\mathfrak{A}$ at Q is in $T_*(G(2,Q \wedge JQ),Q)$, and thus Q must be a critical point of r.

The $\mathbf{Z}_2$-cohomology of $\eta(\mathfrak{A})/_{\xi\simeq-\xi} \overset{\text{def}}{=} \bar{\mathfrak{A}}$ will now yield lower bounds on the number of distinct nonholomorphic critical planes of r_R, $R \in \mathscr{K}(3)^+$. First, as $\eta(\mathfrak{A})$ is a 2-sphere bundle over $\mathbf{C}P^2$, the Serre spectral sequence yields

$$H^*(\eta(\mathfrak{A}), \mathbf{Z}_2) \simeq \begin{cases} \mathbf{Z}_2, & * = 0, 6 \\ \mathbf{Z}_2 \times \mathbf{Z}_2, & * = 2, 4 \\ 0 & * = 1, 3, 5. \end{cases}$$

Then, by [12, p. 145],

$$\begin{aligned} 0 &\to H^0(\bar{\mathfrak{A}}) \to \mathbf{Z}_2 \to H^0(\bar{\mathfrak{A}}) \to H^1(\bar{\mathfrak{A}}) \to 0 \to H^1(\mathfrak{A}) \to \dots \to H^5(\bar{\mathfrak{A}}) \\ &\to H^6(\bar{\mathfrak{A}}) \to \mathbf{Z}_2 \to H^6(\bar{\mathfrak{A}}) \to 0 \end{aligned}$$

is exact. If $h_i = \operatorname{rank}_{\mathbf{Z}_2}(H^i(\bar{\mathfrak{A}}))$, this sequence and Poincaré duality implies that $h_i = h_{6-i}$, $h_0 = 1$, $h_1 = 1$, $h_3 \leq h_2$, and $3 = 2h_2 - h_3$. Either $h_3 = 1$, $h_2 = 2$, or $h_3 = h_2 = 3$. By an example in the next section the first case holds, thus

$\sum_i \operatorname{rank}(H_i(\bar{\mathfrak{A}}, \mathbf{Z}_2)) = 9$. The arguments of Proposition (3.3) then yield

PROPOSITION 3.8. *If $R \in \mathscr{K}(3)^+$, and if r_R, $r_{R|\mathfrak{A}}$ are nondegenerate, then r_R has at least* 3 *distinct holomorphic critical planes, and* 9 *distinct nonholomorphic critical planes.*

Remark. As in Theorem (3.2), those $R \in \mathscr{K}(3)^+$ satisfying these nondegeneracy conditions contain an open dense subset of $\mathscr{K}(3)^+$.

4. A NORMAL FORM THEOREM

The purpose of this section is to establish a normal form theorem for generic $R \in \mathscr{K}(n)^+$, $n = 2, 3$. The following example clearly illustrates the need to specialize to Kähler operators in this paper: in the real case the existence of a normal form is a strong assumption on the operator and would seem to be false for a general $R \in \mathscr{R}(n)$, $n > 3$.

EXAMPLE 4.1. *There are $R \in \mathscr{R}(4)$ that do not have normal forms relative to $\mathscr{R}(4)$.*

Define R, with respect to an orthonormal basis $\{v_i\}$ of $\mathbf{R}^4$, by $R_{1212} = -R_{3434} = 3$, $R_{1313} = -R_{2424} = 2$, $R_{1414} = -R_{2323} = 1$, all other terms 0. This R is in the invariant subspace of trace $-$ 0 operators with zero Weyl tensor [17], thus $R = T \wedge (Id)$, $T =$ Ricci tensor of R, where $T \wedge (Id) \in \mathscr{R}(n)$ is defined by

$$T \wedge (Id)(v \wedge w) = (1/(n-2))(Tv \wedge w + v \wedge Tw - (1/(n-1))(\text{trace}(T))\, v \wedge w)$$

for $T \in \text{Sym}(\mathbf{R}^4)$ [10], [17]. Note that each v_i is an eigenvector of T.

Claim. If $v \wedge w = P$ is a critical plane, then v, w may be chosen to be eignevectors of T.

Proof. Corollary (2.3) and the above decomposition of R imply that Tv, Tw are in the span of $\{v, w\}$. By choosing v, w to diagonalize the restriction of T to P the claim follows.

Thus $\{v_i \wedge v_j\}$ are the only critical planes of R. To see that this P does not have a normal form, define $\mathscr{K} \in \mathscr{R}(4)$ with respect to the same basis by $K_{1234} = K_{3412} = 1$, $K_{2314} = K_{1423} = -1$, all other terms zero. Then $R + \lambda K$ is proper for all λ, and for λ near 0 (as r_R is nondegenerate) $R + \lambda K$ will have only $v_i \wedge v_j$ as critical planes, with the same critical values.

A similar example has been established by S. Zoltek, using somewhat different methods [20].

THEOREM 4.2. *If $n = 2, 3$, then $\{R \in \mathscr{K}(n)^+ \mid R$ has a normal form relative to $\mathscr{K}(n)\}$ contains an open dense subset of $\mathscr{K}(n)^+$.*

Remark. By an elementary but complicated argument this result may be extended considerably in dimension 4. In fact, any $R \in \mathscr{K}(2)$ has a normal form relative to $\mathscr{K}(2)$ [7]. Note also that these normal forms may not be unique (cf. Lemma (4.4)).

Proof. For any $P \in G(2, 2n)$, let $M(P)\colon \mathscr{K}(n) \to V \otimes_{\mathbf{R}} \mathbf{C}$ be defined by $M(P)(R) = R(P)(P)$ as in Corollary (2.3). Note that $M(P)(R) = -iAP$ if and only if P is a critical plane of r_R with critical value A. Define a linear map $M(P_1, \ldots, P_k)\colon \mathscr{K}(n) \to (V \otimes \mathbf{C})^k$ by

$$M(P_1, \ldots, P_k)(R) = (M(P_1)(R), \ldots, M(P_k)(R)).$$

If $P_1, \ldots, P_k$ is a set of distinct critical planes of R, and if R' has the same critical planes and the same critical values, then $R - R' \in \ker(M(P_1, \ldots, P_k))$.

Thus, to show that $\{(P_i, A_i)\}_{i=1,\dots,k}$ is a normal form for R, it suffices to show that $M(P_1, \dots, P_k)_{|\mathscr{S}^\perp}$ is injective.

The proof of this theorem is a "modeling". A normal form for an example is carried over to almost all other $R \in \mathscr{K}(n)^+$. The theorem will only be verified in the case $n = 3$, the case $n = 2$ is similar except at one point, where the proper path will be outlined.

LEMMA 4.3. *There is an $R \in \mathscr{K}(3)^+$ with precisely 3 distinct holomorphic, 9 distinct nonholomorphic critical planes, achieving the lower bounds in* Section 3. *Furthermore, R is nondegenerate, and R has a normal form. In fact, R has 2 distinct types of normal form.*

Proof. First let $S = \sigma(T)$ (cf. Section 1), where T is positive-definite with 3 distinct eigenvalues λ_1, λ_2, λ_3. These eigenvalues will be picked to satisfy genericity conditions below. Assume for definiteness that $\lambda_1 + \lambda_2 + \lambda_3 = 3$. If $T = Id$, then $S_0 = \lambda R_0$ (section 1), and for this operator the space $\mathfrak{A}$ (section 3) is $I^\perp \cap G(2,6)$. Thus, if each λ_i is chosen close enough to 1, $\mathfrak{A}$ for S will be very near $I^\perp \cap G$. Choose λ_i so that, if $v \wedge w \in \mathfrak{A}$, $|\langle v, Jw\rangle| < (1/3)$.

Let $v \wedge w$ be a critical point of r_S, with critical value A. Choose v, w in $v \wedge w$ so that $\langle Tv, w\rangle = 0$ by diagonalizing the restriction of T to $v \wedge w$. Similarly to Example (4.1), by Corollary (2.3) and Proposition (1.6) we obtain

$$\begin{aligned} Aw = S(v \wedge w)v &= (1/10)[(\langle Tv, v\rangle - (3/2))w + Tw - 3\langle v, Jw\rangle JTv \\ &\quad - (3\langle Tv, Jw\rangle + (3/2)\langle v, Jw\rangle)Jv], \\ Av = (S(w \wedge v)w &= (1/12)[(\langle Tw, w\rangle - (3/2))v + Tv + 3\langle v, Jw\rangle JTw \\ &\quad + (3\langle Tv, Jw\rangle + (3/2)\langle v, Jw\rangle)Jw] \end{aligned}$$

Solving for Tv and Tw, as $1 - 9\langle v, Jw\rangle^2 \neq 0$, Tv, $Tw \in \text{span}\{v, Jv, w, Jw\}$. An eigenvector of the restriction of T to this span is thus an eigenvector of T, hence, if $\{v_\alpha\}$ is a unitary basis of V consisting of eigenvectors of T with v_i, v_{i*} the λ_i-eigenvectors,

$$v \wedge w \in \Lambda^2(\text{span}\{v_i, v_{i*}, v_j, v_{j*}\}) \overset{\text{def}}{=} \Lambda^2(i,j)$$

for some i, j. $v \wedge w$ is clearly a critical point of $r_{S_{|G \cap \Lambda^2(i,j)}}$. If $v \wedge w$ is holomorphic, it is clear that $v = v_i$, $w = v_{i*}$. If $w \neq Jv$, either

$$v \wedge w \in \text{span}\{v_i \wedge v_j, v_i \wedge v_{j*}, v_j \wedge v_{i*}, v_{i*} \wedge v_{j*}\}$$

or $\eta(v \wedge w) = a_{ij} v_i \wedge v_{i*} + b_{ij} v_j \wedge v_{j*}$ with

$$a_{ij}(S_{ii*ii*} + S_{ii*jj*}) + b_{ij}(S_{ii*jj*} + S_{jj*jj*}) = 0,$$

determining a_{ij}, b_{ij} up to sign. For generic choices of the λ_i, the critical points $v_i \wedge v_{i*}$, and any Q_{ij} so that $\eta(Q_{ij}) = a_{ij v_i} \wedge v_{i*} + b_{ij} v_j \wedge v_{j*}$ are nondegenerate, and $\eta(\text{span}\{v_i \wedge v_j, \dots, v_{i*} \wedge v_{j*}\} \cap G)$ is a nondegenerate critical circle of $\bar{r}_S$.

Now perturb S to R by $R_{ijij} = S_{ijij} + \varepsilon_{ij}$; $R_{ij*ij*} = S_{ij*ij*} - \varepsilon_{ij}$, for ε_{ij} small. For generic choices of the ε_{ij}, the points $v_i \wedge v_j$, $v_i \wedge v_{j*}$ will be nondegenerate critical points of r_R. To show R is nondegenerate it now suffices to prove there are no other critical planes. By continuity, the only additional critical planes Q would be such that $\eta(Q)$ is in the critical circle of $\bar{r}_S$, as Q must necessarily lie within a tubular neighborhood of this circle for ε_{ij} sufficiently small, but $\bar{r}_R$ restricted to a fiber of this neighborhood has critical point only at the intersection with the circle. However, the only critical points that $\bar{r}_R$ has on this circle are $\eta(v_i \wedge v_j)$ and $\eta(v_i \wedge v_{j*})$. Note that these critical planes achieve the bounds claimed.

To show that, for generic choices of λ_i, this R has a normal form is not difficult. Consider first the critical points $\{v_i \wedge v_{i*}, v_i \wedge v_j, v_i \wedge v_{j*}\}_{i,j=1,2,3}$. If

$$K \in \ker(M(v_{\alpha} \wedge v_{\beta}))$$

for all α, β, and K proper, the first Bianchi identity and Proposition (2.2) easily yield $K = 0$. Thus these critical planes and their associated critical values constitute a normal form for R. Secondly, consider the 9 distinct nonholomorphic planes $\{Q_{ij}, v_i \wedge v_j, v_i \wedge v_{j*}\}$. $\ker(M(Q_{ij}, v_i \wedge v_j, v_i \wedge v_{j*})_{|\mathscr{S}^{\perp}}) \neq 0$ only if

$$a_{23} b_{13} a_{12} - b_{23} a_{13} b_{12} = 0.$$

But a generic choice of the λ_i's will prevent this.

Now assume that R satisfies the hypotheses of the theorem but does not have a normal form. It suffices to find R' arbitrarily near R with a normal form. By Theorem (3.2), and remarks following that result, we may assume that r_R, $r_{R|\mathbf{C}P^{n-1}}$, and $\bar{r}_{R|\eta(\mathfrak{A})}$ are each nondegenerate. Note that $\{R \mid r_R$ degenerate$\}$ contains, as an open dense subset, $\{R \mid r_R$ has only one degenerate critical point, with r_{**} of nullity 1$\}$. The same remark applies to $r_{|\mathbf{C}P^{n-1}}$, $\bar{r}_{|\eta(\mathfrak{A})}$. These claims are established similarly to Theorem (3.2). Let aR° be a multiple of the operator R° given in Lemma (4.3), where $a > 0$ is large enough so that $r_{aR^{\circ}} > r_R$, which is possible as $r_{R^{\circ}} > 0$. By perturbing R if need be, assume that the path

$$t \to (1-t)\,aR^{\circ} + tR = R_t$$

meets $\{R \mid r_R$ (etc.) degenerate$\}$ in a finite number of points t_j, $j = 1, \ldots, \ell$, for which r_R has only one degenerate critical point at which r_{R**} and whichever of $(r_{|\mathbf{C}P^{n-1}})_{**}$ or $(\bar{r}_{|\eta(\mathfrak{A})})_{**}$ applies have nullity one.

LEMMA 4.4. *If $t_0 \in (0,1)$ is such that r_{t_0} is nondegenerate, and if for some neighborhood $(t_0 - \varepsilon, t_0)$ R_t has a normal form for all but finitely many $t \in (t_0 - \varepsilon, t_0)$ of the form (P_{it}, A_{it}), P_{it} holomorphic for* i $< a$, *nonholomorphic for* i $\geq a$, *then there is a $\delta > 0$ so that R_t has a normal form of the same type for all but finitely many $t \in (t_0 - \varepsilon, t_0 + \delta)$, with the curves P_{it} real-analytic.*

Proof. Let $i < a$. If $P_{i0} = P_{it_0}$, as P_{i0} is holomorphic, so is P_{it} for t near t_0, as r_{**} does not change rank. Let $U \subset G$ be a real-analytic coordinate neighborhood of P_{i0}, and let τ: $T^*(U) \to \mathbf{R}^8$ be an analytic coordinate trivialization of $T^*(U)$, that is, τ is the projection onto the fiber. Define ϕ: $U \times I \to \mathbf{R}^8$ by $\phi(P,t) = \tau(dr_t(P))$, where $r_t = r_{R_t}$. By definition of the hessian,

$$d(\phi_{|U\times t_0})(P_{i0}, t_0) \simeq (r_{t_0})_{**}(P_{i0}).$$

Note that $\phi(P_{i0}, t_0) = 0$. The implicit function theorem then implies that there is a neighborhood V of t_0 and $t \to P_{it}$, $t \in V$, so that $\phi(P_{it}, t) = 0$. As ϕ is real-analytic, so is P_{it} [5]. $\phi(P_{it}, t) = 0$ if and only if P_{it} is a critical plane. Note that P_{it} is holomorphic. A similar argument applies to $\bar{r}$ if P_{it} is nonholomorphic, and to $r_{|\mathbf{C}P^{n-1}}$ (though this necessarily yields the same curve P_{it}).

The normal form statement follows by the analyticity in t of the condition $\ker(M(\{P_{it}\})_{|\mathscr{S}}) = 0$.

LEMMA 4.5. *If there is a t_0 with r_t degenerate, with degenerate critical point P_{i0}, and if for all but finitely many $t \in (t_0 - \varepsilon, t_0)$ (resp., $t \in (t_0, t_0 + \varepsilon)$), R_t has a normal form $\{(P_{kt}, A_{kt})\}$, then there is an analytic function $t = t(s)$ ($t(0) = t_0$), s in a neighborhood V of 0 in $\mathbf{R}$, and a regular, analytic curve P_{is} so that $\{(P_{is}, A_{is}), (P_{kt(s)}, A_{kt(s)}) \mid k \neq i\}$ is a normal form for all but finitely many $s \in V$.*

Proof. Assume that P_{i0} is holomorphic, with r_{**} of nullity one. Let $\gamma\colon \mathbf{R} \to \mathscr{K}(3)$ be defined by $\gamma(t) = (1 - t_0 + t^2)aR^\circ + (t_0 - t^2)R$, reparametrizing R_t, $t \leq t_0$ (resp., for the $t > t_0$ case, $\gamma(t) = (1 - t_0 - t^2)aR^\circ + (t_0 - t^2)R$). Let $U \times V \subset G \times I$ be an analytic product neighborhood of (P_{i0}, t_0) with coordinate functions $\{x_i, t\}$, and let $\xi \in T^*(G \times I, (P_{i0}, t_0))$ be chosen transverse to $(\mathrm{Image}(r_{**}(P_{i0})) + \{\lambda dt\}$, ξ *not* orthogonal to dt. Let $s\colon U \times V \to \mathbf{R}$ be analytic and such that $ds(P_{i0}, t_0) = \xi$. Using the coordinate system $\{x_i, s\}$ to trivialize $U \times V$, let $\tau\colon T^*(U \times V) \to \mathbf{R}^9$ be the analytic coordinate trivialization of $T^*(U \times V)$ (the projection onto the fiber) given by this coordinate system. Define $\phi\colon U \times V \to \mathbf{R}^8$ by

$$\phi(P, t) = \pi_s(\tau(dr(P, t))),$$

where $r(P, t) = r_{\gamma(t-t_0)}(P)$ and π_s is the linear projection $\pi_s(\tau(ds)) = 0$, $\pi_s(\tau(dx_i)) = \tau(dx_i)$. By the choice of a above, the implicit function theorem applies to this function, yielding the required analytic curves P_{is}. The nonholomorphic case is similar. The remainder of this lemma proceeds as in the previous lemma.

These two lemmas imply that, for the homotopy $H\colon G \times I \to \mathbf{R}$ given by $H(P, t) = r_t(P)$, the critical points of $H_{|G\times\{t\}}$ form real-analytic smooth arcs in $G \times I$ (called *critical curves*). Furthermore, the endpoints of these curves can only occur on $\partial(G \times I)$. If those curves beginning at the critical points $(P_i, 0)$ of $H_{|G\times\{0\}} = r_{aR^\circ}$ also meet $G \times \{1\}$, the analyticity in t of the condition

$$\ker(M(\{P_{it}\})_{|\mathscr{S}^\perp}) = 0$$

will imply that, for all but finitely many $t \in I$, R_t will have a normal form of either of the types described in Lemma (4.3), by pushing the normal form of aR° along these critical curves.

As $r_0 = r_{R^\circ}$ is nondegenerate and has a minimal number of critical points for such functions, that the critical curves meet $G \times \{1\}$ follows from

LEMMA 4.6. *Let M be a compact Riemannian manifold of dimension greater* *than or equal to* 3. *Let* $H: M \times I \to \mathbf{R}$ *be a smooth homotopy with* $H_{|M\times\{t\}}$ *a Morse function except for finitely many* $t_i \neq 0, 1$ *for which* $H_{|M\times\{t_i\}}$ *has only one degenerate critical point of nullity one. Assume moreover that the critical curves of* H *are smooth, regular curves. If there is a critical curve* γ of H *beginning at* $(x, 0)$ *such that* γ *does not meet* $M \times \{1\}$, *then there is a Morse function* $g\colon M \to R$ *with fewer critical points than* $H_{|M\times\{0\}}$.

Proof. As $\dim(M) \geq 3$, perturb H so that, if $\pi\colon M \times I \to M$ is the product projection, $\pi(\gamma_i)$ are disjoint smooth curves in M without double points, where $\{\gamma_i\}$ are the critical curves of H. Let γ be as stated, and let V be a tubular neighborhood of $\pi(\gamma)$ missing the other $\pi(\gamma_i)$. Note that V is contractible, and contains two critical points of $H_{|M\times\{0\}}$. There is a $t_0 < 1$ such that $\gamma \subset M \times [0, t_0)$, so there is a smooth function $t = t(x)$, $t\colon M \to [0, t_0]$ with

(i) $t_{|M-V} = 0$,
(ii) $M' = \text{graph}(t) \subset M \times I$ is contained in $(M - V) \times \{0\} \cup V \times [0, t_0]$, and (iii) the interior of $M' \cup M \times \{0\}$ contains γ.

Construct a vector field X in M using M'. Denote by $\sim$ the duality between 1-forms and vector fields on M; that is, $\tilde{X}(Y) = \langle X, Y\rangle$ for X, Y vector fields. Give $M \times I$ the product metric. Define X by $X(m) = \pi_*(d\tilde{H}(m, t(m)))$. Note that $X_{|V}$ is nowhere zero, as no critical points of $H_{|M\times\{t\}}$ are on M' by construction. It is now elementary to show that $\tilde{X}$ is closed, hence X is a gradient field as $\tilde{X}$ is exact, being cohomologous to $d(H_{|M\times\{0\}})$. Thus $\tilde{X} = dg$. g has critical points only at zeros of X, that is, at all critical points of $H_{|M\times\{0\}}$ except the two endpoints of γ. g is still a Morse function as the behavior of g near a critical point not in V is the same as that of $H_{|M\times\{0\}}$.

This lemma may be applied directly to $r_{|\mathbf{C}P^{n-1}}$ and $r_{|\mathfrak{A}}$ in the case $n = 3$, completing the proof of the theorem in that case. In the case $n = 2$ the statement of the lemma must be modified to accomodate the situation where $H_{|M\times\{t\}}$ is not Morse, but has only nondegenerate critical manifolds [4], as $\dim(\mathbf{C}P^1) = 2$. This modification is easily accomplished, and the lemma may then be applied to r itself.

The following corollary is similarly verified:

COROLLARY 4.7. *For* $n = 2, 3$, $\{R \in \mathscr{K}(n)^+$ *the critical points of* r_R *span* $\Lambda^2(V)\}$ *contains an open dense subset of* $\mathscr{K}(n)^+$.

Remark. By suitable choice of model, this corollary may be extended to $\mathscr{K}(n)^+$, $n > 3$, and to $\mathscr{R}(n)$.

5. SPECIAL CASES AND APPLICATIONS

As mentioned in the Introduction, many special properties of the manifold M, or of the algebraic operator R, can be characterized in terms of the critical behavior of the sectional curvature. This section describes the critical behavior of r_R if R is in an invariant subspace of $\mathscr{K}(n)$ (except for the subspace $\mathscr{W}$ when $n \geq 3$) and derives special normal forms for these subspaces if $n = 2$ or 3. Also, the critical behavior of the sectional curvature of certain homogeneous spaces is described, again yielding normal forms of a special type for the Kähler case in low dimensions.

This first proposition follows easily from Corollary (2.3) and the characterization of $\mathscr{I} \oplus \mathscr{B}$ in Proposition (1.6).

PROPOSITION 5.1. *If $R \in \mathscr{I} \oplus \mathscr{B} \subseteq \mathscr{K}(n)$, r_R has critical points $v_j \wedge v_{j*}$, $v_j \wedge v_k$, and $v_j \wedge v_{k*}$ for all v_j, v_k in some unitary basis $\{v_\alpha\}$ of V. If $n = 2$ or 3, these critical points and their critical values are a normal form for R relative to $\mathscr{K}(n)$.*

For $R \in \mathscr{I} \oplus \mathscr{W} = \mathscr{E} \subset \mathscr{K}(2)$, [17] shows that R has a normal form relative to $\mathscr{R}(4)$. The normal form is of a quite special type, following from the fact that $r_R({*}P) = r_R(P)$. Thus, in the Kähler case, for $R \in \mathscr{E}$, if $P = v_1 \wedge v_{1*}$ is critical, then so is $P = v_2 \wedge v_{2*}$, with the same critical values. Moreover, $v_1 \wedge v_2$ and $v_1 \wedge v_{2*}$ are also critical, for an appropriate choice of $\{v_\alpha\}$. These points clearly yield a normal form. There is also a necessary and sufficient condition for $R \in \mathscr{K}(2)$ to be Einstein that depends only on the configuration of the holomorphic critical planes themselves, without reference to the critical values.

THEOREM 5.2. *Let $R \in \mathscr{K}(2)$ be proper.*

(i) *If R has a normal form $\{(P_i, A_i)\}$ where $\langle P_1, P_2\rangle$ are holomorphic, with $\langle P_1, P_2\rangle = 0$ and $A_1 = A_2$, then $R \in \mathscr{E}$. Conversely, if $R \in \mathscr{E}$, such a normal form exists.*

(ii) *R is Einstein if and only if there are holomorphic critical planes P_1, P_2, and P_3 such that $\langle P_1, P_2\rangle = 0$, $\langle P_1, P_3\rangle = 1/2 = \langle P_2, P_3\rangle$.*

Proof. (i) is straightforward. For (ii), first assume that P_1, P_2, P_3 exist as above. A unitary basis may be chosen so that $P_1 = v_1 \wedge v_{1*}$, $P_2 = v_2 \wedge v_{2*}$, and $P_3 = 1/2\,(v_1 + v_2) \wedge (v_{1*} + v_{2*})$. As P_1, P_2 are critical, $R_{ii*i\alpha} = 0$ for $\alpha \neq i^*$. Proposition (2.2) applied to P_3 then yields $R_{11*11*} = R_{22*22*}$, so that $R \in \mathscr{E}$ by (i). To show the converse, choose $\{v_\alpha\}$ so that $P_1 = v_1 \wedge v_{1*}$, $P_2 = v_2 \wedge v_{2*}$, $v_1 \wedge v_2$, and $v_1 \wedge v_{2*}$ are all critical. Then $P_3 = \dfrac{1}{2}(v_1 + v_2) \wedge (v_{1*} + v_{2*})$ will be critical.

If $M = G/U$ is a reductive homogeneous space with a G-invariant metric, [14] gives an explicit formulation for the Riemannian curvature tensor R of M in terms of the Lie algebra $\mathfrak{g}$ of G. In the realm of pointwise geometry—that is, the study of algebraic curvature tensors—there is a general problem yet to be fully answered. As the metric on M is invariant, the curvature tensor lies in a single orbit of $\mathscr{R}(n)$ under $0(n)$. A natural question to ask is which orbits in $\mathscr{R}(n)$ are the curvature tensors of some homogeneous space. The next result gives a partial characterization of those orbits in $\mathscr{R}(n)$ that can be the curvature of a homogeneous space $M = G/U$, where G is compact and rank (U) = rank (G). The characterization is in terms of the critical behavior of the sectional curvature. It should be noted that these are only necessary conditions.

Recall [3, 6] that, under these hypotheses G may be assumed to be semi-simple. $\mathfrak{g}$ can be decomposed by $\mathfrak{g} = \mathfrak{u} + \mathfrak{m}$. If Δ is the set of positive roots of G, then $\mathfrak{g} \simeq \mathfrak{t} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where each $\mathfrak{g}_\alpha = \text{span}\,\{X_\alpha, Y_\alpha\}$ are the (*real*) *root spaces* of G, and $\mathfrak{t}$ is a maximal abelian subalgebra. There is a subset $\Delta' \subseteq \Delta$ such that

$$u \simeq t + \sum_{\alpha \in \Delta'} \mathscr{g}_\alpha;$$

$\Delta'$are called the *roots of* u. $\Delta_c = \Delta - \Delta'$ is the set of *complementary* roots;

$$m \simeq \sum_{\alpha \in \Delta_c} \mathscr{g}_\alpha.$$

Any invariant metric on M will lift to an AdU-invariant positive-definite inner product on m, with $\langle X_\alpha, Y_\alpha \rangle = 0$ and $\langle \mathscr{g}_\alpha, \mathscr{g}_\beta \rangle = 0$ for $\alpha, \beta \in \Delta_c$. These properties depend on the fact that $\text{rank}(U) = \text{rank}(G)$.

THEOREM 5.3. *Let $M = G/U$ be a compact homogeneous space with invariant metric, where G is compact and* $\text{rank}(U) = \text{rank}(G)$. *Let R be the curvature tensor of M at eU.*

(1) *For each complementary root α, the plane $X_\alpha \wedge Y_\alpha / |X_\alpha|^2$ is a critical plane of r_R. Thus there are* dim $(M)/2$ *orthonormal critical planes. In each case the critical value is positive.*

(2) *R maps* $\operatorname{span}_{\alpha,\beta} \{X_\alpha \wedge Y_\beta\}$ *into itself, and both of* $\operatorname{span}_{\alpha,\beta} \{X_\alpha \wedge Y_\beta\}$ *and* $\operatorname{span}_{\alpha,\beta} \{Y_\alpha \wedge Y_\beta\}$ *into* $\operatorname{span}_{\alpha,\beta} \{X_\alpha \wedge X_\beta + Y_\alpha \wedge Y_\beta\}$.

(3) *Any critical point of the restriction of r_R to*

(a) span $\{X_\alpha \wedge X_\beta\} \cap G(2,n)$

(b) span $\{Y_\alpha \wedge Y_\beta\} \cap G(2,n)$ *or*

(c) span $\{X_\alpha \wedge Y_\beta\} \cap G(2,n)$,

is critical point of r_R.

Remark. The spaces (a), (b), and (c) are mutually orthogonal. Also, note that it is not claimed that all critical planes are of this form.

Proof. Using [14], it is easy to see that the operators B: $m \times m \to m$ and L: $m \times m \to m$ defined there satisfy $B(X_\alpha, Y_\alpha) = 0 = B(X_\alpha, X_\alpha) = B(Y_\alpha, Y_\alpha)$, and similarly with L. By Nomizu's expression for R in terms of B and L, (1) follows. (2) may be verified in a similar fashion, and (3) follows from (2) and Proposition (2.2).

If now M is assumed to have an invariant complex structure, it is well-known [2] that M supports a Kähler metric (in fact, there are families of such metrics). Note that, if M is a compact, simply-connected, Kähler homogeneous space, it is necessarily of this type. It is easily seen that the complex structure tensor J, lifted to an operator on m, leaves each $\mathscr{g}_\alpha$ invariant for $\alpha \in \Delta_c$. Moreover, $JX_\alpha = Y_\alpha$, $JY_\alpha = -X_\alpha$ (by choice of a set of positive roots).

The Kähler condition will imply that $\langle X_{\alpha+\beta}, X_{\alpha+\beta} \rangle = \langle X_\alpha, X_\alpha \rangle + \langle X_\beta, X_\beta \rangle$ whenever α, β and $\alpha + \beta$ are complimentary roots. This implies a slight strengthening of Theorem (5.3); $\operatorname{span}_{\alpha} \{X_\alpha \wedge Y_\alpha\}$ is mapped into itself by R, as is $\operatorname{span}_{\alpha \neq \beta} \{X_\alpha \wedge Y_\beta\}$.

Using this, we may obtain normal form theorems in low dimensions.

THEOREM 5.4. *If $n = 2$ or 3, the curvature tensor R of $M = G/U$ a compact, simply-connected Kähler homogeneous space has a normal form relative to $\mathscr{K}(n)$.*

Proof. (for $n = 3$). Let $v_i = X_{\alpha_i}/|X_{\alpha_i}|$ for some ordering $\{\alpha_i\}$ of Δ_c, $i = 1$, 2, 3. Then $v_{i*} = Y_{\alpha_i}/|Y_{\alpha_i}|$. Choose $P_i = v_i \wedge v_{i*}$, and $Q_1, \ldots, Q_6$ so that Q_1, Q_2, and Q_3 are critical points of the restriction of r_R to

$$\operatorname*{span}_{i\neq j} \{v_i \wedge v_j\} \cap G(2,6)$$

and Q_4, Q_5, Q_6 are critical points of the restriction of r_R to

$$\operatorname*{span}_{i\neq j} \{v_i \wedge v_{j*}\} \cap G(2,6).$$

As each of these intersections are 2-spheres, and r_R restricted to each is quadratic, the Q_i may be chosen to be orthonormal. By Theorem (5.3) and the above discussion, each of the P_i and Q_j are critical points of r_R. Note that $Q_j \neq v_\alpha \wedge v_\beta$ unless $\alpha \pm 2\beta$, $\beta \pm 2\alpha$ are not complementary roots. A direct calculation shows that these points and their critical values form a normal form.

Remark. Again using classical decomposition theorems for the curvature tensor, it is easy to show that, if $M \subset \mathbf{C}P^{n+1}$ is a complex hypersurface, its curvature tensor has a normal form relative to $\mathscr{K}(n)$ consisting of orthonormal planes.

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