UNIT VECTOR FIELDS ON ANTIPODALY PUNCTURED SPHERES: BIG INDEX, BIG VOLUME

by Fabiano G.B. Brito, Pablo M. Chacón & David L. Johnson

Abstract. — We establish in this paper a lower bound for the volume of a unit vector field $\vec{v}$ defined on $\mathbf{S}^n \setminus \{\pm x\}$, $n = 2, 3$. This lower bound is related to the sum of the absolute values of the indices of $\vec{v}$ at x and $-x$.

Résumé (*Champs unitaires dans les sphères antipodalement trouées: grand indice entraïne grand volume.*)

Nous etablissons une borne inferieure pour le volume d'un champ de vecteurs $\vec{v}$ defini dans $\mathbf{S}^n \setminus \{\pm x\}$, $n = 2, 3$. Cette borne inferieure depend de la somme des valeurs absolus des indices de $\vec{v}$ en x et en $-x$.

1. Introduction

The volume of a unit vector field $\vec{v}$ on a closed Riemannian manifold M is defined [8] as the volume of the section $\vec{v} : M \to T^1M$, where the Sasakian metric is considered in T^1M. The volume of $\vec{v}$ can be computed from the

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Levi-Civita connection ∇ of M. For an orthonormal local frame $\{e_a\}_{a=1}^n$, we have:

$$\begin{aligned}\mathrm{vol}(\vec{v}) = \int_M \Bigg(1 + \sum_{a=1}^{n} \|\nabla_{e_a}\vec{v}\|^2 + \sum_{a_1<a_2} \|\nabla_{e_{a_1}}\vec{v} \wedge \nabla_{e_{a_2}}\vec{v}\|^2 + \dots \\ \dots + \sum_{a_1<\cdots<a_{n-1}} \|\nabla_{e_{a_1}}\vec{v} \wedge \cdots \wedge \nabla_{e_{a_{n-1}}}\vec{v}\|^2\Bigg)^{\frac{1}{2}}.\end{aligned} \tag{1}$$

Note that $\mathrm{vol}(\vec{v}) \geq \mathrm{vol}(M)$ and also that only parallel fields attain the trivial minimum.

For odd dimensional spheres, vector fields homologous to the Hopf fibration have been studied, see [8], [3], [7] and [1]. In [5], a non-trivial lower bound of the volume of unit vector fields on spaces of constant curvature was obtained. In $\mathbf{S}^{2k+1}$, only the vector field tangent to the geodesics from a fixed point (with two singularities) attains the volume of that bound. We notice that unit vector fields with singularities show up in a natural way, see also [9].

For manifolds of dimension 5, a theorem showing how the topology of a vector field influences its volume appears in [4]. More precisely, the result in [4] is an inequality relating the volume of $\vec{v}$ and the Euler form of the orthogonal distribution to $\vec{v}$.

The purpose of this paper is to establish a relationship between the volume of unit vector fields and the indices of those fields around isolated singularities.

We consider these notes to be a preliminary effort to understand this phenomenon. For this reason, we have chosen a simple model where such a relationship is found. We hope this could serve as inspiration for more complex situations to be treated in a near future.

Precisely, we prove here:

THEOREM 1.1. — *Let $W = \boldsymbol{S}^n \setminus \{N, S\}$, $n = 2$ or 3, be the standard Euclidean sphere where two antipodal points N and S are removed. Let $\vec{v}$ be a unit smooth vector field defined on W. Then,*

$$\text{for } n = 2, \qquad \mathrm{vol}(\vec{v}) \geq \Big(\pi + |I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2\Big)\frac{\mathrm{vol}(\boldsymbol{S}^2)}{2};$$

$$\text{for } n = 3, \qquad \mathrm{vol}(\vec{v}) \geq \Big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)|\Big)\mathrm{vol}(\boldsymbol{S}^3),$$

where $I_{\vec{v}}(P)$ stands the Poincaré index of $\vec{v}$ around P.

We will comment briefly some possible extensions for this result in section 3 of this paper.

2. Proof of the Theorem

Even though there is a common line of reasoning in the proof of both parts of the Theorem, each dimension has its special features. For that reason, we provide separate proofs for dimensions 2 and 3.

2.1. Case $n = 2$. — Denote by g the usual metric on $\mathbf{S}^2$ induced from $\mathbb{R}^3$. Without loss of generality we take $N = (0,0,1)$ and $S = (0,0,-1)$. On W we consider an oriented orthonormal local frame $\{e_1, e_2 = \vec{v}\}$. Its dual basis is denoted by $\{\theta_1, \theta_2\}$ and the connection 1-forms of ∇ are $\omega_{ij}(X) = g(\nabla_X e_j, e_i)$ for $i,j = 1,2$ where X is a vector in the corresponding tangent space. In dimension 2, the volume (1) reduces to:

$$\mathrm{vol}(\vec{v}) = \int_{\mathbf{S}^2} \sqrt{1 + k^2 + \tau^2},$$

where $k = g(\nabla_{\vec{v}}\vec{v}, e_1)$ is the geodesic curvature of the integral curves of $\vec{v}$ and $\tau = g(\nabla_{e_1}\vec{v}, e_1)$ is the geodesic curvature of the curves orthogonal to $\vec{v}$. Also,

$$\omega_{12} = \tau\theta_1 + k\theta_2.$$

The first goal is to relate the integrand of the volume with the connection form ω_{12}. If S^1_φ is the parallel of $\mathbf{S}^2$ at latitude $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ consider the unit field $\vec{u}$ on S^1_φ such that $\{\vec{u}, \vec{n}\}$ is positively oriented where $\vec{n}$ is the field pointing toward N. Let $\alpha \in [0, 2\pi]$ be the oriented angle from $\vec{u}$ to $\vec{v}$. Then $\vec{u} = \sin\alpha e_1 + \cos\alpha \vec{v}$. If $i : S^1_\alpha \to \mathbf{S}^2$ is the inclusion map, we have:

$$i^*\omega_{12}(\vec{u}) = \tau\theta_1(\vec{u}) + k\theta_2(\vec{u}) = \tau\sin\alpha + k\cos\alpha. \tag{2}$$

We split the domain of the integral in northern and southern hemisphere, H^+ and H^- respectively. First we consider the northern hemisphere H^+. From the general inequality $\sqrt{a^2+b^2} \geq |a\cos\beta + b\sin\beta| \geq a\cos\beta + b\sin\beta$, for any $a, b, \beta \in \mathbb{R}$, we have:

$$\begin{aligned} \sqrt{1+k^2+\tau^2} &\geq \cos\varphi + \sqrt{k^2+\tau^2}\sin\varphi \\ &\geq \cos\varphi + \left|k\cos\alpha + \tau\sin\alpha\right|\sin\varphi \\ &\geq \cos\varphi + \left|i^*\omega_{12}\right|\sin\varphi. \end{aligned} \tag{3}$$

From (2) and (3) we get:

$$\begin{aligned} \mathrm{vol}(\vec{v})\big|_{H^+} &\geq \int_{H^+} \left(\cos\varphi + \left|i^*\omega_{12}\right|\sin\varphi\right) \\ &= \int_0^{\frac{\pi}{2}} \int_{S^1_\varphi} \cos\varphi + \int_0^{\frac{\pi}{2}} \int_{S^1_\varphi} \left|i^*\omega_{12}\right|\sin\varphi \\ &\geq \int_0^{\frac{\pi}{2}} 2\pi\cos^2\varphi + \int_0^{\frac{\pi}{2}} \sin\varphi \left|\int_{S^1_\varphi} i^*\omega_{12}\right|. \end{aligned} \tag{4}$$

The connection form ω_{12} fulfills $d\omega_{12} = \theta_1 \wedge \theta_2$. Therefore, the area of the annulus region $A(\varphi, \frac{\pi}{2} - \epsilon) = \{(x_1, x_2, x_3) \in \mathbf{S}^2 | \sin\varphi \leq x_3 \leq \sin(\frac{\pi}{2} - \epsilon)\}$ provides the equality:

$$\begin{aligned}\int_{A(\varphi,\frac{\pi}{2}-\epsilon)} d\omega_{12} = \text{area of } A = & \int_{\varphi}^{\frac{\pi}{2}-\epsilon} 2\pi\cos t \\ =& 2\pi\big(\sin(\frac{\pi}{2} - \epsilon) - \sin\varphi\big).\end{aligned} \tag{5}$$

The border of $A(\varphi, \frac{\pi}{2} - \epsilon)$ is $\partial A = S^1_\varphi \cup S^1_{\frac{\pi}{2}-\epsilon}$ (with the appropriate orientation), so by (5) and Stokes' Theorem:

$$\begin{aligned}\int_{S^1_\varphi} i^*\omega_{12} =& \int_{A(\varphi,\frac{\pi}{2}-\epsilon)} d\omega_{12} + \int_{S^1_{\frac{\pi}{2}-\epsilon}} i^*\omega_{12} \\ =& 2\pi\big(\sin(\frac{\pi}{2} - \epsilon) - \sin\varphi\big) + \int_{S^1_{\frac{\pi}{2}-\epsilon}} i^*\omega_{12}.\end{aligned} \tag{6}$$

If ω is the Riemannian connection form on the principal frame bundle of $T(\mathbf{S}^2)$, which is $T^1(\mathbf{S}^2)$, then the restriction of ω to the vertical directions is the volume form of the fiber. Since $\omega_{12} = \vec{v}^*(\omega)$, the index of $\vec{v}$ at N is:

$$\lim_{\epsilon\to 0} \int_{S^1_{\frac{\pi}{2}-\epsilon}} i^*\omega_{12} = \lim_{\epsilon\to 0} \int_{S^1_{\frac{\pi}{2}-\epsilon}} i^*\vec{v}^*\omega = \mathrm{vol}(\mathbf{S}^1) I_{\vec{v}}(N).$$

Thus, from (6):

$$\int_{S^1_\varphi} i^*\omega_{12} = 2\pi(1 - \sin\varphi) + 2\pi I_{\vec{v}}(N). \tag{7}$$

Following from (4) with (7) we have:

$$\begin{aligned}\mathrm{vol}(\vec{v})\big|_{H^+} \geq& \frac{\pi^2}{2} + \int_0^{\frac{\pi}{2}} \sin\varphi \Big| 2\pi(1 - \sin\varphi) + 2\pi I_{\vec{v}}(N) \Big| \\ =& \frac{\pi^2}{2} + \int_0^{\frac{\pi}{2}} \big| 2\pi\sin\varphi I_{\vec{v}}(N) - 2\pi\sin\varphi(\sin\varphi - 1) \big| \\ \geq& \frac{\pi^2}{2} + \int_0^{\frac{\pi}{2}} \Big| \big|2\pi\sin\varphi I_{\vec{v}}(N)\big| - \big|2\pi\sin\varphi(\sin\varphi - 1)\big| \Big| \\ \geq& \frac{\pi^2}{2} + \Big| \int_0^{\frac{\pi}{2}} \big|2\pi\sin\varphi I_{\vec{v}}(N)\big| - \big|2\pi\sin\varphi(\sin\varphi - 1)\big| \Big| \\ =& \frac{\pi^2}{2} + \Big| 2\pi|I_{\vec{v}}(N)| \int_0^{\frac{\pi}{2}} \sin\varphi - 2\pi \int_0^{\frac{\pi}{2}} (\sin\varphi - \sin^2\varphi) \Big| \\ =& \frac{\pi^2}{2} + \Big| 2\pi|I_{\vec{v}}(N)| - 2\pi + \frac{\pi^2}{2} \Big|.\end{aligned} \tag{8}$$

For the southern hemisphere, similarly to (5) and (6) with $-\frac{\pi}{2} < -\frac{\pi}{2}+\epsilon < \varphi \leq 0$:

$$\begin{aligned}\int_{S^1_\varphi} i^*\omega_{12} &= \int_{S^1_{-\frac{\pi}{2}+\epsilon}} i^*\omega_{12} - \int_{A(-\frac{\pi}{2}+\epsilon,\varphi)} d\omega_{12}\\ &= \int_{S^1_{-\frac{\pi}{2}+\epsilon}} i^*\omega_{12} - 2\pi\big(\sin\varphi - \sin(-\frac{\pi}{2}+\epsilon)\big).\end{aligned}$$

Now the index of $\vec{v}$ at S is obtained by $\lim\limits_{\epsilon\to 0}\int_{S^1_{-\frac{\pi}{2}+\epsilon}} i^*\omega_{12} = \mathrm{vol}(\mathbf{S}^1) I_{\vec{v}}(S)$. Therefore:

$$\int_{S^1_\varphi} i^*\omega_{12} = 2\pi I_{\vec{v}}(S) - 2\pi(\sin\varphi + 1). \tag{9}$$

In order to obtain a similar equation to (3) we take $\beta = -\varphi$, and together with (2) we have:

$$\begin{aligned}\mathrm{vol}(\vec{v})\big|_{H^-} &\geq \int_{H^-}\big(\cos\varphi - \big|i^*(\omega_{12})\big|\sin\varphi\big)\\ &\geq \int_{-\frac{\pi}{2}}^{0} 2\pi\cos^2\varphi - \int_{-\frac{\pi}{2}}^{0}\bigg|\int_{S^1_\varphi} i^*\omega_{12}\bigg|\sin\varphi.\end{aligned} \tag{10}$$

From (9) and (10):

$$\begin{aligned}\mathrm{vol}(\vec{v})\big|_{H^-} \geq& \frac{\pi^2}{2} - \int_{-\frac{\pi}{2}}^{0}\Big|2\pi I_{\vec{v}}(S) - 2\pi(\sin\varphi+1)\Big|\sin\varphi\\ =&\frac{\pi^2}{2} + \int_{-\frac{\pi}{2}}^{0}\Big|2\pi I_{\vec{v}}(S)\sin\varphi - 2\pi(\sin\varphi+1)\sin\varphi\Big|\\ \geq&\frac{\pi^2}{2} + \int_{-\frac{\pi}{2}}^{0}\Big||2\pi I_{\vec{v}}(S)\sin\varphi| - |2\pi(\sin\varphi+1)\sin\varphi|\Big|\\ \geq&\frac{\pi^2}{2} + \bigg|2\pi|I_{\vec{v}}(S)|\int_{-\frac{\pi}{2}}^{0}|\sin\varphi| - 2\pi\int_{-\frac{\pi}{2}}^{0}|\sin^2\varphi + \sin\varphi|\bigg|\\ =&\frac{\pi^2}{2} + \Big|2\pi|I_{\vec{v}}(S)| - 2\pi + \frac{\pi^2}{2}\Big|.\end{aligned} \tag{11}$$

Finally, recall that the sum of the indices of a field in $\mathbf{S}^2$ must be 2, therefore the sum of the absolute values of the indices must be greater or equal than 2.

So, from (8) and (11), the volume of $\vec{v}$ is bounded by:

$$\begin{aligned}
\mathrm{vol}(\vec{v}) \geq & \pi^2 + \left|2\pi|I_{\vec{v}}(N)| - 2\pi + \frac{\pi^2}{2}\right| + \left|2\pi|I_{\vec{v}}(S)| - 2\pi + \frac{\pi^2}{2}\right| \\
\geq & \pi^2 + \left|2\pi|I_{\vec{v}}(N)| + 2\pi|I_{\vec{v}}(S)| - 4\pi + \pi^2\right| \\
= & \pi^2 + \left|2\pi(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2) + \pi^2\right| \\
= & 2\pi^2 + 2\pi(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2) \\
= & \Big(\pi + |I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2\Big)\frac{\mathrm{vol}(\mathbf{S}^2)}{2}.
\end{aligned}$$

2.2. Case $n = 3$. — As before, denote by g the metric in $\mathbf{S}^3$ and consider a general situation where $N = (0,0,0,1)$, $S = (0,0,0,-1)$ and $I_{\vec{v}}(N) \geq 0$ (and therefore $I_{\vec{v}}(S) \leq 0$).

If $\vec{v}$ is a unit vector field on W, consider on W an oriented orthonormal local frame such that $\{e_1, e_2, e_3 = \vec{v}\}$. The dual basis will be denoted by $\{\theta_1, \theta_2, \theta_3\}$. The coefficients of the second fundamental form of the orthogonal distribution to $\vec{v}$, possibly non-integrable, are $h_{ij} = \omega_{i3}(e_j) = g(\nabla_{e_j}\vec{v}, e_i)$. The coefficients of the acceleration of $\vec{v}$ are given by $\nabla_{\vec{v}}\vec{v} = a_1 e_1 + a_2 e_2$. Finishing the notation, we will use J for the integrand of the volume (1) and:

$$\sigma_2 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}, \quad \sigma_{2,1} = \begin{vmatrix} h_{11} & a_1 \\ h_{21} & a_2 \end{vmatrix}, \quad \sigma_{2,2} = \begin{vmatrix} a_1 & h_{12} \\ a_2 & h_{22} \end{vmatrix}.$$

It is easy to see that:

$$J = \left(1 + \sum_{i,j=1}^{2} h_{ij}^2 + a_1^2 + a_2^2 + \sigma_2^2 + (\sigma_{2,1})^2 + (\sigma_{2,2})^2\right)^{\frac{1}{2}}.$$

Note that $(1 + |\sigma_2|)^2 = 1 + 2|\sigma_2| + \sigma_2^2 \leq 1 + \sum_{i,j=1}^{2} h_{ij}^2 + \sigma_2^2$. Therefore:

$$J \geq \sqrt{(1 + |\sigma_2|)^2 + |\sigma_{2,1}|^2}, \tag{12}$$

where equality holds if and only if $a_1 = a_2 = 0$ and we have either $h_{11} = h_{22}$ and $h_{12} = -h_{21}$, or $h_{11} = -h_{22}$ and $h_{12} = h_{21}$.

Now we want to identify the last term in (12) with the evaluation of certain forms.

In the frame $\{e_1, e_2, \vec{v}\}$ we can demand that e_1 will be tangent to S^2_φ, the parallel of $\mathbf{S}^3$ with latitude $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We complete a frame in S^2_φ with $\vec{u}$ in such a way $\{e_1, \vec{u}\}$ is an oriented local frame compatible with the normal field $\vec{n}$ that points toward the North Pole. That is, in such a a way that $\{e_1, \vec{u}, \vec{n}\}$ is a positively oriented local frame of $\mathbf{S}^3$. Let $\alpha \in [0, 2\pi]$ be the oriented angle from

TS^2_φ to $\vec{v}$ and $i : S^2_\varphi \to \mathbf{S}^3$ the inclusion map. In this way, $\vec{u} = \cos\alpha\vec{v} + \sin\alpha e_2$ and:

$$i^*(\theta_1 \wedge \theta_2)(e_1, \vec{u}) = \sin\alpha,$$
$$i^*(\theta_1 \wedge \theta_3)(e_1, \vec{u}) = \cos\alpha,$$
$$i^*(\theta_2 \wedge \theta_3)(e_1, \vec{u}) = 0.$$

In order to evaluate $i^*(\omega_{13} \wedge \omega_{23})$, first we notice that:

$$\omega_{13} \wedge \omega_{23} = \sigma_2\theta_1 \wedge \theta_2 + \sigma_{2,1}\theta_1 \wedge \theta_3 + \sigma_{2,2}\theta_2 \wedge \theta_3.$$

So, $i^*(\omega_{13} \wedge \omega_{23})(e_1, \vec{u}) = \sin\alpha\sigma_2 + \cos\alpha\sigma_{2,1}$.

As in (3) with $\beta \in [0, \frac{\pi}{2}]$ such that $\sin\beta = |\sin\alpha|$ and $\cos\beta = |\cos\alpha|$, from (12) we get:

$$\begin{aligned} J \geq& \sin\beta(1 + |\sigma_2|) + \cos\beta|\sigma_{2,1}| \\ =&|\sin\alpha| + |\sin\alpha||\sigma_2| + |\cos\alpha||\sigma_{2,1}| \\ \geq&|\sin\alpha| + |\sin\alpha\sigma_2 + \cos\alpha\sigma_{2,1}| \\ =&|i^*(\theta_1 \wedge \theta_2)| + |i^*(\omega_{13} \wedge \omega_{23})| \geq |i^*(\theta_1 \wedge \theta_2) + i^*(\omega_{13} \wedge \omega_{23})|. \end{aligned} \tag{13}$$

We split W in northern and southern hemisphere, H^+ and H^- respectively. Then, from (13):

$$\begin{aligned} \mathrm{vol}(\vec{v})|_{H^+} &\geq \int_{H^+} |i^*(\theta_1 \wedge \theta_2) + i^*(\omega_{13} \wedge \omega_{23})| \\ &\geq \int_0^{\frac{\pi}{2}} \left| \int_{S^2_\varphi} i^*(\theta_1 \wedge \theta_2) + i^*(\omega_{13} \wedge \omega_{23}) \right|. \end{aligned} \tag{14}$$

We know that $d\omega_{12} = \omega_{13} \wedge \omega_{23} + \theta_1 \wedge \theta_2$. If $A(\varphi, \frac{\pi}{2} - \epsilon)$ is the annulus region between the parallels S^2_φ and $S^2_{\frac{\pi}{2}-\epsilon}$, $0 \leq \varphi < \frac{\pi}{2} - \epsilon < \frac{\pi}{2}$, we have by Stokes' Theorem:

$$\int_{S^2_\varphi} i^*(\omega_{13} \wedge \omega_{23}) + i^*(\theta_1 \wedge \theta_2) = \int_{S^2_{\frac{\pi}{2}-\epsilon}} i^*(\omega_{13} \wedge \omega_{23}) + \int_{S^2_{\frac{\pi}{2}-\epsilon}} i^*(\theta_1 \wedge \theta_2). \tag{15}$$

We bound $i^*(\theta_1 \wedge \theta_2) = \sin\alpha \geq -1$ on $S^2_{\frac{\pi}{2}-\epsilon}$ and consequently:

$$\int_{S^2_\varphi} i^*(\omega_{13} \wedge \omega_{23}) + i^*(\theta_1 \wedge \theta_2) \geq \int_{S^2_{\frac{\pi}{2}-\epsilon}} i^*(\omega_{13} \wedge \omega_{23}) - 4\pi\cos^2(\frac{\pi}{2} - \epsilon).$$

As for the 2-dimensional case, $\omega_{13} \wedge \omega_{23}$ is the pull-back under $\vec{v}^*$ of a 2-form ω on $T^1(\mathbf{S}^3)$ which restricts to the volume form on the fibers. In this case, however, the form on the unit tangent bundle is not exactly the connection form, but a wedge of components of the connection form which restricts to the volume form on the fibers of the unit tangent bundle. As before, though, the index of $\vec{v}$ at the singularity N is:

$$\lim_{\epsilon\to 0}\int_{S^2_{\frac{\pi}{2}-\epsilon}} i^*(\omega_{13}\wedge\omega_{23}) = \lim_{\epsilon\to 0}\int_{S^2_{\frac{\pi}{2}-\epsilon}} i^*\vec{v}^*(\omega) = \mathrm{vol}(\mathbf{S}^2)I_{\vec{v}}(N).$$

So,

$$\int_{S^2_\varphi} i^*(\omega_{13}\wedge\omega_{23}) + i^*(\theta_1\wedge\theta_2) \geq 4\pi I_{\vec{v}}(N) \geq 0. \tag{16}$$

From (14) and (16) we get:

$$\mathrm{vol}(\vec{v})|_{H^+} \geq \int_0^{\frac{\pi}{2}} 4\pi|I_{\vec{v}}(N)| = 2\pi^2|I_{\vec{v}}(N)|. \tag{17}$$

In a similar way for the southern hemisphere, the integral of $d\omega_{12}$ over the annulus region $A(-\frac{\pi}{2}+\epsilon,\varphi)$, $-\frac{\pi}{2} < -\frac{\pi}{2}+\epsilon < \varphi \leq 0$ provides exactly (15) but now we bound $\sin\alpha \leq 1$ to obtain:

$$\int_{S^2_\varphi} i^*(\omega_{13}\wedge\omega_{23}) + i^*(\theta_1\wedge\theta_2) \leq \int_{S^2_{-\frac{\pi}{2}+\epsilon}} i^*(\omega_{13}\wedge\omega_{23}) + 4\pi\cos^2(-\frac{\pi}{2}+\epsilon).$$

The index of $\vec{v}$ at S can be calculated as $\lim\limits_{\epsilon\to 0}\int_{S^2_{-\frac{\pi}{2}+\epsilon}} i^*(\omega_{13}\wedge\omega_{23}) = \mathrm{vol}(\mathbf{S}^2)I_{\vec{v}}(S)$. So,

$$\int_{S^2_\varphi} i^*(\omega_{13}\wedge\omega_{23}) + i^*(\theta_1\wedge\theta_2) \leq 4\pi I_{\vec{v}}(S) \leq 0.$$

Therefore,

$$\begin{aligned}\mathrm{vol}(\vec{v})|_{H^-} &\geq \int_{-\frac{\pi}{2}}^0 \left|\int_{S^2_\varphi} i^*(\theta_1\wedge\theta_2) + i^*(\omega_{13}\wedge\omega_{23})\right| \\ &\geq \int_{-\frac{\pi}{2}}^0 4\pi|I_{\vec{v}}(S)| = 2\pi^2|I_{\vec{v}}(S)|.\end{aligned} \tag{18}$$

Thus, from (17) and (18) we have:

$$\mathrm{vol}(\vec{v}) \geq 2\pi^2\big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)|\big) = \big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)|\big)\mathrm{vol}(\mathbf{S}^3).$$

3. Concluding Remarks

It is easy to verify that the north-south field $\vec{n}$ achieves the equalities in the main Theorem. In fact, the volume of $\vec{n}$ in $\mathbf{S}^2$ is equal to $\frac{\pi}{2}\mathrm{vol}(\mathbf{S}^2)$, and in $\mathbf{S}^3$ is $2\mathrm{vol}(\mathbf{S}^3)$. The lower bound in $\mathbf{S}^3$ when the singularities are trivial (i.e. $I_{\vec{v}}(N) = I_{\vec{v}}(S) = 0$) has no special meaning.

These results should extend to higher dimensions if one makes use of some rather complicated inequalities involving the volume integrand in (1) of a unit vector field and some symmetric functions coming from the second fundamental form of the orthogonal distribution (which is generally non integrable). Some of these inequalities can be found in [6] or [5].

Index results should exist also for the case when the spheres are punctured differently. In other words, if we have two singularities which are not antipodal points of $\mathbf{S}^2$ or $\mathbf{S}^3$ or if we have more than two singularities, what could be said? We believe that some results relating indices and positions of the singularities to the volume of a unit vector field may be found.

For singular vector fields on $\mathbf{S}^2$ another natural situation is the one of unit vector fields defined on $\mathbf{S}^2 \setminus \{x\}$. In a recent paper [2], see also [9], a unit vector field $\vec{p}$ is defined on $\mathbf{S}^2 \setminus \{x\}$ by parallel translation of a given tangent vector at $-x$ along the minimizing geodesics to x. It has been proved in [2] that $\vec{p}$ minimizes the volume of unit vector fields defined on $\mathbf{S}^2 \setminus \{x\}$. By a direct calculation, we obtain the inequality $\mathrm{vol}(\vec{p}) > \mathrm{vol}(\vec{n})$, where $\vec{n}$ is the north-south vector field tangent to the longitudes of W.

Now, new questions arise about minimality on specific topological-geometrical configurations on the punctured spheres.

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