1 Work

Another use of calculus is to calculate the amount of work done completing a task. For basic things, we have the formula

$$\text{Work} = \text{Force} \times \text{Distance}$$

Example 1.1. Do an example in metric and Imperial.

But just like with our formulas for velocity and such, this only works if the force and the distance are both constants. But in real life, they aren't always. Often, the force can change as work is done. Let's say that at a given point x the force is $f(x)$. How do we calculate the work done to move it from point a to point b? Well, we do this like we've done everything else so far. We approximate.

First, we notice that even though the force can change over the interval, if we break it up into smaller intervals, it might change as much. So first we break the interval down into subintervals $[x_i, x_{i+1}]$ and look at the work down over the smaller interval. The force will still change, but the formula $W = Fd$ will be a better approximation. The distance of each interval is Δx. For the work done, pick any point in the interval, say the right endpoint. This gives us

$$W \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

Just like before, we let the number of intervals go to infinity and this the volume.

Definition 1.

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \int_a^b f(x)dx \end{aligned}$$

Example 1.2. When a particle is located x feet away form the origin, a force of $x^3 + 7x$ acts on it. How much is work done in moving it from $x = 1$ to $x = 3$?

A somewhat more useful example is a spring. There is something called Hooke's law which says that force required to keep a string stretched from its resting length by x (if x isn't too large) is

$$f(x) = kx$$

k is a positive constant and depends on the spring.

Example 1.3. The work required to stretch a spring from its resting length of 10 cm to 15 cm is 40 N. How much work is done in stretching it from 15 cm to 18 cm?

*Remember that even though Hooke's law works for any units, Newtons are $kg \cdot m/s^2$, so everything in here needs to be in meters.

Example 1.4. A chain on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?

Example 1.5. 6.4 # 17

Example 1.6. Conical tank: 10 m tall, 4 m radius, 2 m from the top.

2 Inverses

Let's say you have a chart that has the population of the country at given times. This is a function (a numerical representation). Now, a question we might ask is, "If we're given a population, when did this occur?" This is a question about the inverse.

Not all functions have inverses. The only ones that do are what are called one-to-one.

Definition 2. A function is called one-to-one if it never takes on the same value twice. In other words, if $x \neq y$ then $f(x) \neq f(y)$.

That seems a little weird. An easy way to test, though, is the horizontal line test. If any horizontal line you draw only intersects the graph once, then the function is one-to-one.

Example 2.1. $y = x^2$, $y = x^3$

Armed with this knowledge, we can define the inverse of a function.

Definition 3. Let f be one-to-one with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by:

$$\text{If } f(x) = y, \text{ then } f^{-1}(y) = x \text{ and vice versa.}$$

*Draw the arrow diagram.

Example 2.2. $y = x^3$

$$\text{The -1 in } f^{-1} \text{ is not an exponent}$$
$$f^{-1}(x) \text{ is not } \frac{1}{f(x)}$$

Based off of the definition, we get two nice equations:

$$f^{-1}\left(f(x)\right) = x$$

$$f\left(f^{-1}(y)\right) = y$$

How do you find an inverse, however?

1. First, write $y = f(x)$.
2. Solve this equation for x in terms of y (so you want $x = g(y)$)
3. Interchange the x and y. So $f^{-1}(x) = g(x)$.

Example 2.3. $f(x) = x^3 + 2$

But what about the graph? The graph of f^-1 is obtained by reflecting the graph of f about the line $y = x$.

2.1 Calculus with Inverses

For most of what we do here, we'll need this:

Theorem 1. If f is one-to-one and continuous, then f^{-1} is also continuous.

But what about differentiability? Well, it's not necessarily differentiable because of the following theorem:

Theorem 2. If f is one-to-one and differentiable with inverse $g = f^{-1}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and

$$g'(a) = \frac{1}{f'(g(a))}$$

Example 2.4. $y = x^2$, $x = 1$

Example 2.5. $f(x) = 2x + \cos x$, $x = 1$