1 Indefinite Integrals

The fundamental theorem of calculus shows just how important antiderivatives are. Since we'll be using them so frequently from now on, we introduce a notation for them. Actually, the FTC gives us an intuitive notation for them.

Definition 1. $\int f(x)dx$ is an antiderivative of $f(x)$. This is called the indefinite integral.

*Do a few examples

It is incredibly important to remember the difference between a definite and an indefinite integral. $\int_a^b f(x)dx$ is a number and x shouldn't appear anywhere. $\int f(x)dx$ is a family of functions and should be nothing but x's.

Here's a quick reminder of antidifferentiation formulas.

*Put up a table

Example 1.1.

$$\int (18x^5 - 3\sec x \tan x)dx$$

Example 1.2.

$$\int \frac{\sin\theta}{\cos^2\theta}dx$$

*Do a few examples of definite integrals.

Finally, recall how I was talking about how the definite integral is a net area or a displacement. Well, another way to state the fundamental theorem of calculus is that the integral of the rate of change is the net change. *Talk about displacement and then some other examples.

Example 1.3. $v(t) = t^2 - t - 6$. Find the displacement and the distance travelled.

*Do plenty more examples.

2 Substitution Rule

So far, we know the FTC says we can use antiderivatives to do definite integrals. The problem is that we can't really do a lot of integrals. Just like we learned techniques to do complicated derivatives, we have techniques to do complicated antiderivatives. In fact, the two major techniques we have are inverses of two rules for differentiation: the chain rule and the product rule.

The first rule we'll do is the substitution rule, which is the chain rule backwards.

Theorem 1. If $u = g(x)$ is a differentiable function whose range is an interval $[a, b]$ and f is continuous on $[a, b]$, then

$$\int f(g(x))\, g'(x)dx = \int f(u)du$$

Now, it seems kind of weird to have that du since u is actually function, but the substitution rule says that's just fine. (Actually, that's not bizarre at all. dx is just du where $u = g(x) = x$.)

But how do we use this?

Example 2.1.

$$\int x^3\cos(x^4 + 2)dx$$

So what you do is look for something - anything - in the function that you can differentiate and get something else in the integrand to cancel it out with. As a tip, if there's a constant added or subtracted in there, always include it. It won't change the du.

However, sometimes it's not so obvious what you want to differentiate

Example 2.2.

$$\int \sqrt{2x+1}dx$$

*Work some more examples.

2.1 Definite Integrals

Here's how this rule applies to definite integrals.

Theorem 2. If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

What this says that when you do a substitution rule, you change the limits of integration. See, the a and b are the limits for x. They represent the values that x takes on. But once you have du, you need the values that u takes on, not just x.

Example 2.3.

$$\int_0^4 \sqrt{2x+1}dx$$

Remember this rule: When you're plugging in values, the limits have to match the variable!

2.2 Symmetry

The substitution rule gives us some nice ways to deal with even and odd functions:

Theorem 3. If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)dx = 0$.

*Do a few examples
*Work more examples of the substitution rule.

3 Area Between Curves

Up until now, we've been finding the (net) area beneath curves. What about if we want the area between two curves? Well, we can find that, too.

*Draw an example along with rectangles.

So, if we let the number of rectangles go to infinity, we get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

But this should look familiar. This is just the integral of $f(x) - g(x)$.

Theorem 4. The area A of a region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$ is

$$A = \int_a^b (f(x) - g(x))\, dx$$

You can also see this because it's the area under f minus the area under g. If one of them becomes negative, the area becomes negative and so you're adding areas together.

Example 3.1. Find the area between $y = x^2 + 1$ and $y = x$ between $x = 0$ and $x = 1$

However, we don't always need bounds on the left and right. Often, these bounds will be determined by the equations themselves.

Example 3.2. Find the area enclosed by $y = x^2$ and $y = 2x - x^2$.

Sometimes, this becomes even more complicated and the functions can cross over each other. This gives us a more general form for area between curves.

Example 3.3.

$$A = \int_a^b |f(x) - g(x)|\, dx$$

This just says that when you're drawing your rectangles, the height is determined by subtracting the smaller from the larger. In this case bigger=top and smaller=bottom.

Example 3.4. Find the area bounded by $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/2$

Always remember: Areas are positive. Always do large minus small when computing areas and you'll be okay.

Sometimes we don't want to look at vertical rectangles.

Example 3.5. Find the area bounded by $y = x - 1$ and $y^2 = 2x + 6$.

When doing horizontal rectangles, it's the same idea. Bigger minus smaller. With these, though, bigger=right and smaller=left.

*Do a few more examples.