1 Distances (again)

Example 1.1. Speedometer readings for a motorcycle at 12 second intervals are given:

$t(s)$	0	12	24	36	48	60
v	30	28	25	22	24	27

Give two different estimates.

2 Definite Integral

Recall what we just did.

Definition 1. If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 = 1, x_1, \ldots, x_n = b$ be the endpoints of the intervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these intervals, so x_i^* is in $[x_{i-1}, x_i]$. The definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

To make things easier, we usually pick the same point in every interval, usually either the left endpoint or the right endpoint. So usually $x_i^* = x_i$ or $x_i^* = x_{i-1}$. Also, as long as f is continuous, it doesn't matter which one we pick. They'll always end up the same.

Definition 2. In $\int_a^b f(x)dx$, the $f(x)$ is called the integrand. The a and b are limits of integration - a is the lower limit and b is the upper limit. dx has no official meaning by itself. It's pretty much just a symbol. Sort of.

One important thing I want you to remember: $\int_a^b f(x)dx$ is a number and doesn't depend on x. Any variable works without changing the value. Also, since $\int_a^b f(x)dx$ is a number, there should be no variables in there. Ever. If your answer has an x in it, it's wrong.

This was introduced in terms of areas and distances. And if a function is always positive, that's exactly what it is. However, if the function has negative values, this means more of a net area instead of area or a displacement instead of distance.

*Draw an example

Example 2.1.

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\tan(x_i^2) - \sqrt{x_i})\Delta x$$

From 0 to π

2.1 Computing Integrals

In order to compute these, we'll need the following rules:

1. $\sum_{i=1}^{n} c = nc$
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
3. $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
5. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

7. $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 2.2.

$$\int_0^3 (x^3 - 6x)dx$$

Keep in mind, an integral is still an area, even if it is just a net area.

Example 2.3.

$$\int_0^3 (x - 1)dx$$

2.2 Properties

It's important to keep in mind the two interpretations of an integral: area and sum. Let's keep thinking of it as a sum and come up with some properties

1. $\int_a^b c dx = c(b - a)$
2. $\int_a^b cf(x)dx = c \int_a^b f(x)dx$
3. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

If we switch a and b, Δx becomes $(a - b)/n$ which is $-(b - a)/n$, so

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Additionally, if $a = b$, then $\Delta x = 0$ and so

$$\int_a^a f(x)dx = 0$$

Example 2.4.

$$\int_0^1 (4 + 3x^2)dx$$

A very nice property is the following:

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Finally, if $a \leq b$,

1. If $f(x) \geq 0$ for x in $[a, b]$, then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x)$ for x in $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
3. If $m \leq f(x) \leq M$ for x in $[a, b]$, then $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$

Example 2.5. Estimate $\int_1^4 \sqrt{x}dx$

Example 2.6.

$$\int_1^4 (x^2 + 2x - 5)dx$$

3 Fundamental Theorem of Calculus

This is possible one of the biggest theorems in calculus, period. It links both differential and integral calculus and shows that they're both pretty much the same problem. Instead of just differential calculus being time travel, all of calculus is time travel. Calc I and II are the DeLorean. Limits, are still like Mr. Fusion - but now so are derivatives and integrals. The Fundamental Theorem of Calculus is the Flux Capacitor. It's what makes all of this possible.

It's so great that it's actually in two parts, called part 1 and part 2. The first part deals with functions like

$$g(x) = \int_a^x f(t)dt$$

Example 3.1. *Draw an example (hopefully one that's not so difficult to compute.

Fundamental Theorem of Calculus, Part 1 1. If f is continuous on $[a, b]$, then the function

$$g(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.

Example 3.2. Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}dt$

Even though it looks weird, remember that this is still a normal derivative.

Example 3.3.

$$\frac{d}{dx}\int_1^{x^4} \sec t dt$$

Now for the other half of the FTC. The first part was how to differentiate functions, so this one is how to integrate them. This is the one that will show up pretty much every day for the rest of the course. It's usefulness cannot be exaggerated.

Fundamental Theorem of Calculus, Part 2 1. If f is continuous on $[a, b]$ and F is <u>any</u> antiderivative of f ($F' = f$), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 3.4.

$$\int_0^3 (x^3 - 6x)dx$$

Example 3.5.

$$\frac{d}{dx}\left(\int_1^x (2+t^4)^5 dt\right)$$

Example 3.6.

$$\frac{d}{dx}\left(\int_1^{\cos x} (t + \sin t)dt\right)$$

Example 3.7.

$$\int_1^8 \sqrt[3]{x}dx$$

Example 3.8.

$$\int_\pi^{2\pi} \cos\theta d\theta$$