1 Review

- Limit laws
- Derivative rules

2 Antiderivatives

Recall when we were looking at the motion of a particle, we were given a function for its position at a given time. We could figure out how fast it was moving by looking at the derivative. What if we had the velocity, but wanted the position? Well, velocity is the derivative of the position, so if we could find a function that has a derivative of the velocity we're looking for that'd be the position.

Definition 1. A function F is an antiderivative of f in an interval $[a, b]$ if $F'(x) = f(x)$ for all x in $[a, b]$.

Example 2.1. $f(x) = x^2$. It's not hard to see that an antiderivative is $F(x) = \frac{1}{3}x^3$. But what about $G(x) = \frac{1}{3}x^3 + 1$?

A consequence of the mean value theorem is that if two functions have the same derivative on an interval, then they differ by a constant. So if G and F are both antiderivatives of f, then $F(x) = G(x) + C$.

Theorem 1. If F is an antiderivative of f on an interval $[a, b]$, then the most general antiderivative of f on I is:

$$F(x) + C$$

Where C is an arbitrary constant.

So if we go back to $f(x) = x^2$, the most general antiderivative is $\frac{1}{3}x^3 + C$.

Example 2.2.

1. $f(x) = \sin x$
2. $f(x) = x^n$, $n \geq 0$
3. $f(x) = x^{-3}$

Here are some useful formulas. Also, it's standard notation for capital letters to be antiderivatives of the lowercase ones, i.e. $F' = f$ and $G' = g$.

Function	Antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
x^n if $n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

Example 2.3. $f'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$

A lot of the time, however, we don't want that $+C$ floating around. We can get rid of it, though, if we have more data.

Example 2.4. $f'(x) = x\sqrt{x}$, $f(1) = 2$.

Now, let's go back to the motion of a particle.

Example 2.5. $a(t) = 6t + 4$, $v(0) = -6\text{cm/s}$ and $s(0) = 9$).

We can also do slightly (very slightly) more interesting problems.

Example 2.6. A ball is thrown upward with a speed of 15 m/s from the edge of a cliff 100 meters above the ground. Find its height after t seconds. When does it reach its maximum height? When does it hit the ground?

3 Areas and Distances

3.1 Areas

Let's say we want to find the following area

*Draw a curve

How would we do this? For polygons, we have nice formulas. Even for other shapes, we have ways of computing the area. But what about this? Well, for arbitrary polygons, we can break it up into triangles and use those to compute the area. We can actually modify that idea and use it here.

Example 3.1. $f(x) = x^2$, $0 \leq x \leq 1$

Go over rectangles, right endpoints, left endpoints, etc.

n	L_n	R_n
10	0.285	0.385
100	0.32835	0.33835
1000	0.3328335	0.3338335

Example 3.2. $\lim\limits_{n\to\infty} R_n = \frac{1}{3}$

Use $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

You can also do the same thing for L_n. Now lets use this on the original function I put up:

*Do the drawing again.

Definition 2. The area, A, of a region, S, that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} (f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x)$$

A similar definition works for L_n, except the indices go from 0 to $n-1$. However, it doesn't matter which point in the interval we picked to draw the rectangles. Any one of them would work, so

$$A = \lim_{n\to\infty} (f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x)$$

is a more general definition.

This is a lot to write, so we'll use sigma notation:

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

So to rewrite the definitions, we have

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

Example 3.3. $f(x) = \cos x$, $0 \leq x \leq \pi/2$

3.2 Distances

Say we know how fast something is going at a particular time. We can use this to find out the distance it travelled. If the distance is constant, we have $d = rt$. If it's not, we have to use more sophisticated tools.

Example 3.4. Velocity of a car is:

$Time$	$Velocity$
0	25
5	30
10	35
15	40
20	45
25	45
30	40

This seems similar to finding areas because it's the same idea. Remember how velocity was the same as the slope of a tangent line? Well, the distance is the same as the area.

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t$$