1 Related Rates

So far we've been looking at the rate of change of one "thing". Maybe it's a particle or just a function or what have you. A lot of the time, however, when something changes it causes something else to change. For instance, say you're pumping air into a balloon. The volume, radius, and surface area of the balloon all change. The thing is, we can relate all of these quantities so we can relate the rate of change.

This method lets us find the rate of change of some quantity if we have the rate of change of something else. Another example is two cars travelling in different directions. Their individual rate of change relates to the rate of change of the distance between the two cars.

There are two basic steps in this kind of problem:

1. Find an equation that relates the quantities in the problem.

2. Use implicit differentiation to get the rate of change that you want.

Example 1.1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the radius is 25 cm?

Example 1.2. A 10 ft long ladder leans against a wall. The bottom of the ladder slides away from the wall at a rate of 1 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall? How fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/4$?

Example 1.3. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the light?

Example 1.4. §3.9 # 33

Example 1.5. §3.9 # 37

2 Linear Approximations and Differentials

*Draw a graph and a tangent line and show how the two are close together

Sometimes, we can compute the value of a function at a, but not anywhere near a. Since the tangent is close to the function near a, we can use the tangent line.

Recall: The formula for the tangent line at $x = a$ is $y = f'(a)(x - a) + f(a)$. Define the linear approximation to be $L(x) = f(a) + f'(a)(x - a)$.

So now, if we know $f(a)$ and $f'(a)$ we can approximate $f(a)$ close to a.

Example 2.1. Linearize $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Example 2.2. Linearize $f(x) = \sin x$ and $g(x) = \cos x$ around $a = 1$

2.1 Differentials

Differentials are the same thing as doing linear approximations, but using different words. The differential of a function is dy and it's defined as $dy = f'(x)dx$.

Δy and Δx represent the actual changes in y and x, resp. But we can get the change in y corresponding to a change in x exactly. It's just $\Delta y = f(x + \Delta x) - f(x)$

*Draw tangent line to show relationship of dy and Δy and dx and Δx.

So the change in y can be approximated by dy. Thus we say that $f(a + dx) \approx f(a) + dy$. But this is just different notation for the exact same procedure.