# 1 Chain Rule

Recall how we took the derivatives of combined functions using addition, subtraction, division, and multiplication. However, when we first learned about combining functions, we had one last method: composition. There's a rule for that, too. It's the chain rule. However, this one is more complicated but also far more useful, so it's treated separately.

**Chain Rule.** If $f$ and $g$ are both differentiable and $F = f \circ g$ is the composition of the functions (recall: $F(x) = f(g(x))$), then $F$ is differentiable and $F'$ is given by the product

$$F'(x) = f'(g(x))g'(x)$$

In the othe notation, let $y = f(u)$ and $u = g(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

*Use the blah description here.

Let's try an example

**Example 1.1.** Find $F'(x)$ is $F(x) = \sqrt{x^2+1}$.

*For this, use both notations. Emphasize taking derivatives with respect to a variable.

The chain rule is like the flux capacitor - it makes taking derivatives possible. Sure, it has help from other components like Mr. Fusion, but it's the flux capacitor that really does it.

Also important is that we work from inside out.

**Example 1.2.** Find $F'(x)$ and $G'(x)$ if $F(x) = \sin^2 x$ and $G(x) = \sin(x^2)$

Let's go back to the Leibniz notation. If we let $y = \sin u$ as above, then $\frac{dy}{dx} = \frac{d}{dx}(\sin u) = \frac{dy}{du}\frac{d}{dx} = \cos u \frac{du}{dx}$. So derivative of sin of blah wrt $x$ is cos of blah times the derivative of blah.

We can use the same idea to get formulas from the formulas we already have. Most of them aren't very useful to remember, but one that shows up all the time is the power rule.

**Proposition 1.** If $n$ is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Or

$$\frac{d}{dx}\left(g(x)\right)^n = n\left(g(x)\right)^{n-1} g'(x)$$

**Example 1.3.**

| | | |
|---|---|---|
| $y = (x^3-1)^{100}$ | $y = \frac{1}{\sqrt{3x^2+x+1}}$ | $y = \left(\frac{x-2}{2x+1}\right)^9$ |
| $y = (2x+1)^5(x^3-x+1)^4$ | $y = \sin(\cos(\tan(x)))$ | $y = \sqrt{\sec(x^3)}$ |

# 2 Implicit Differentiation

There's another use for the chain rule. Remember when we define lines, we can define them by an equation like $3x + 7y = 2$. If we want to find $\frac{dy}{dx}$, we can just solve for $y$ and then compute it. However, we can define other functions like this. For insance, a circle with radius 1 is given by $x^2 + y^2 = 1$. If we want fo find $\frac{dy}{dx}$ we can still solve for $y$, but we have to be careful about signs. But we can really get as funky as we want with this and get an equation where you simply can't solve for $y$. However, we can still find $\frac{dy}{dx}$.

**Example 2.1.** $x^2 + y^2 = 25$. Find tangent line at (3,4).

*Emphasize that you don't always need to get rid of the $y$ in the equation for $\frac{dy}{dx}$.

Here's one where it's impossible to find $y$

**Example 2.2.** $\sin(x+y) = y^2 \cos x$ Find $\frac{dy}{dx}$

Do other examples here:

# 3 Higher Derivatives

Remember how I mentioned that the derivative is a function? That's what let's us get all these nice formulas. Everything we can do with a function we can do with a function. Well, one thing we do with functions is take the derivative. So, if $f'$ is a function, then we write the derivative as $(f')' = f''$. This is the second derivative. In Leibniz notation, it's $\frac{d^2y}{dx^2}$

**Example 3.1.** $f(x) = x\cos x$. Find $f''(x)$

What does the second derivative mean?

**Example 3.2.** $s(t) = t^3 - 6t^2 + 9t$

But if the derivative is a function, then $f''$ is a function and we can take the derivative and get the third derivative, $f'''$. We can keep going with the fourth derivitave $f''''$ and so on. But after the second derivative, they're mostly written either as $f^{(n)}$ or using Leibniz notation $\frac{d^ny}{dx}^n$

The third derivative is sometimes called jerk.

**Example 3.3.** $y = x^3 - 6x^2 - 5x + 3$

Let's find a generic $n$th derivative.

**Example 3.4.** $f(x) = \frac{1}{x}$. Find $f^{(n)}(x)$

*Introduce $n!$.

We can also do this implicitly

**Example 3.5.** Find $y''$ if $x^4 + y^4 = 16$