1 Formulas

Remember, mathematics is all about being lazy. Using the limit definition of the derivative will get very difficult if we have to do it every time. Fortunately, we have methods for computing common derivatives like we have methods for computing common limits.

$\frac{d}{dx}(c)$

Prove this.

$\frac{d}{dx}(x) = 1$

Prove this.

$\frac{d}{dx}(x^n) = nx^{n-1}$

Prove this using the binomial theorem.

1.1 Derivative laws (like limit laws)

- $\frac{d}{dx}\left(cf(x)\right)$
- $\frac{d}{dx}\left(f(x) + g(x)\right)$
- $\frac{d}{dx}\left(f(x) - g(x)\right)$

  Use scalar multiplication + addition.

Example 1.1. Find $f'(x)$ if $f(x) = x^8 + 7x^5 - 20x^3 + 17x^2 - 1$

Example 1.2. Find the points of $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

1.2 Product and Quotient rules

- Product Rule

  Pull out and $f(x + h)$ and a $g(x)$.

- Quotient Rule

  Add and subtract $f(x)g(x)$

Example 1.3. Find $f'(x)$ if $f(x) = (6x^3)(7x^4)$

Example 1.4. Find $\frac{d}{dx}\left(\frac{x^2+x-2}{x^3+6}\right)$

1.3 Power Functions

Prove $\frac{d}{dx}\left(x^{-n}\right) = -n^{-n-1}$ using quotient rule.

Example 1.5. $f(x) = \frac{1}{x}$, $f(x) = \frac{6}{t^3}$

Example 1.6. Recall what happens if $f(x) = \sqrt{x}$

State, but don't prove $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real n.

Example 1.7. At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

2 Trig Functions

So far, the formulas we have can be used to find derivatves for polynomials, rational functions, and some more general algebraic functions (but not all, yet). However, what about transcendental functions? Well, exponentials and logarithms will be saved for a later day. Trig functions, however, we can take care of now.

Let's start off with $f(x) = \sin x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos h \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos h \sin h}{h} \right) \\ &= \lim_{h\to 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin x}{h} \right) \right) \end{aligned}$$

We can break it up and compute $\lim_{h\to 0} \sin x = \sin x$ and $\lim_{h\to 0} \cos x = \cos x$ easily enough. But what about $\lim_{h\to 0} \frac{\cos h - 1}{h}$ and $\lim_{h\to 0} \frac{\sin h}{h}$? These limits are tricky since we don't have any rules to compute them.

Use $\sin\theta < \theta$ and $\theta < \tan\theta$ for $\frac{\sin h}{h}$. Use conjugate + trig identity for the cos one.

Put together to get $\frac{d}{dx} \sin x = \cos x$

Now for $\cos x$

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h\to 0} \left(\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right) \\ &= \lim_{h\to 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) \\ &= \sin x \end{aligned}$$

So that's sin and cos. What about tangent?. For this, we use the formula $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule.

Example 2.1. $\frac{d}{dx}(\sec x)$

Example 2.2. $\frac{d}{dx}\left(\frac{\sec x}{1+\tan x}\right)$

Let's reuse the limit we used to get the formulas.

If there's time, do this:

Example 2.3. $\lim_{x\to 0} \frac{\sin 7x}{4x}$

3 Some Applications

Recall how a derivative can be used