1 Tangents and velocities

On the first day, I talked very briefly about where calculus comes from. One branch comes from studying tangent lines to curves. Here's a more in-depth overview. A tangent line to a circle is basic.

(draw picture)

But a tangent line to a curve is far more difficult to see. A tangent to a circle intersects only in one place, but tangents to curves can intersect in more than one place:

(draw picture)

Example 1.1. Use $y = x^2$ and look at difference quotient. Use left and right limits with a chart. Introduce limit notation and show limit process.

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999
1.001	2.001
1.01	2.01
1.1	2.1
1.5	2.5
2	3

Velocity is a problem that might sound different, but turns out to be exactly the same as finding tangent lines.

Example 1.2. Use falling ball problem. Galileo found distance travelled at time t is proportional to the square of the time (ignoring resistance). Use the equation $s(t) = 4.9t^2$

Average velocity is easy, but velocity at a specific moment in time is different. Again, we do approximations. (difference quotient).

Time	Average Velocity
5-6	53.9
5-5.1	49.49
5-5.05	49.245
5-5.01	49.049
5-5.001	49.0049

Again, state limiting process

The second example may seem similar to the first. The reason is because they're the same. First, we draw the graph of $s(t)$. Then we look at the graph of the line at $(a, 4.9a^2)$ with slope equal to the instantaneous velocity and then the tangent line. You can see that they're the same. So in order to compute these, we're going to need to be able to do this limiting process in a better way.

2 Limits

Let's put tangent lines aside for now and just look at taking limits themselves.

2.1 Computing limits

Example 2.1. $f(x) = x^2 - x + 2$

x	$f(x)$	x	$f(x)$
1.99	3.9701	2.01	4.0301
1.995	3.985	2.005	4.015
1.999	3.997	2.001	4.003

We can see that as we get closer to $x = 2$ from the left and the right, $f(x)$ gets closer to 4. We write this as $\lim_{x\to 2}(x^2 - x + 2) = 4$

Definition 1. We write

$$\lim_{x\to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, is L"

if we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a but not equal to a.

Emphasize that $x \neq a$.

Example 2.2.

$$\lim_{x\to 1} \frac{x-1}{x^2-1}$$

x	$f(x)$	x	$f(x)$
0.99	0.5025	1.01	0.4975
0.999	0.50025	1.001	0.49975
0.9999	0.500025	1.0001	0.499975

So $\lim_{x\to 1} \frac{x-1}{x^2-1} = 0.5$

Show what happens by adding $f(x) = 2$ at $x = 1$

However, there are problems using tables to get limits.

Example 2.3.

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$$

Look at $1/n$ and show the problems.

Example 2.4.

$$\lim_{x\to 0}\left(x^3 + \frac{\cos 5x}{10000}\right)$$

1	1.000028
0.5	0.12492
0.1	0.001088
0.05	0.000222
0.01	0.000101

Don't worry, though. We'll figure out how to do this. Now, let's look at another problem that shows up.

Example 2.5.

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Show the limit doesn't exist

Definition 2. We say that

$$\lim_{x \to a^-} f(x) = L$$

or the left-hand limit of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the left) is L if we can make $f(x)$ as close to L as we want by taking x close enough to a and $x < a$.

The right-hand limit is defined in the same way.
Using definitions, we can see that

Proposition 1. $\lim_{x \to a} f(x) = L$ *if and only if* $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Use picture in the book.

2.2 Infinite limits

Example 2.6.

$$\lim_{x \to 0} \frac{1}{x^2}$$

Introduce notation $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Definition 3. Let f be defined on both sides of a but not necessarily defined at a. Then

$$\lim_{x \to a} f(x) = \infty$$

means that we can make $f(x)$ as large as we want by taking x close enough to a, but not equal to a.

Show negative infinity and mention one-sided limits and infinities.

Definition 4. The line $x = a$ is a vertical asymptote of $y = f(x)$ if one of the following is true:

(all the infinite limits go here)

Example 2.7.

$$\lim_{x \to 3^{\pm}} \frac{2x}{x-3}$$

3 Calculating limits

I mentioned that using tables is a poor way to compute limits. Fortunately, we have better methods. First, we're going to need what are called the 5 *Limit Laws*

Proposition 2. *Suppose that c is a constant and that*

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

Then

1. $\lim_{x \to a} (f(x) + g(x)) = L + M$
2. $\lim_{x \to a} (f(x) - g(x)) = L - M$
3. $\lim_{x \to a} (cf(x)) = cL$
4. $\lim_{x \to a} (f(x)g(x)) = LM$
5. $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M}$ *if* $M \neq 0$

Example 3.1. Try this

If we use the product law more than once, we get

Proposition 3. *6.* $\lim_{x \to a} (f(x))^n = (\lim_{x \to a})^n$ *if n is a positive integer.*

To use these laws, we'll need two special limits

Proposition 4. *7.* $\lim_{x \to a} c = c$ *and 8.* $\lim_{x \to a} x = a$

I probably won't prove these, but that will depend on time.

Letting $f(x) = x$ and using 6 and 8, we get

Proposition 5. *9.* $\lim_{x\to a} x^n = a^n$ *if* n *is a positive integer.*

We also have the properties

Proposition 6. *10.* $\lim_{x\to a} x^{1/n} = a^{1/n}$
11. $\lim_{x\to a} (f(x))^{1/n} = (\lim_{x\to a} f(x))^{1/n}$

Using these, we can calculate many different limits.

Example 3.2.

$$\lim_{x\to 5} 2x^2 - 3x + 4$$

$$\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

In both of these cases, if we plugged a into the function, we'd get the limit. This gives us the following

Direct Substitution Property. *If* f *is a polynomial or a rational function and* a *is in the domain of* f*, then*

$$\lim_{x\to a} f(x) = f(a)$$

However, this doesn't work for all functions.

Example 3.3.

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$$

$$\lim_{t\to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

3.1 Using one-sided limits

Example 3.4. $x = 0$ for $f(x) = |x|$ and $f(x) = |x|/x$

Here are two more properties of limits that are quite useful

Proposition 7. *If* $f(x) \le g(x)$ *when* x *is near* a *but not necessarily at* $x = a$*, then* $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

Squeeze Theorem. *If* $f(x) \le g(x) \le h(x)$ *when* x *is near* a *and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Draw a picture

Example 3.5.

$$\lim_{x\to 0} x^2 \sin\frac{1}{x}$$