

# 1 Trigonometry

## 1.1 Angles

I mentioned on Tuesday that we don't use degrees in calculus. Instead we use a unit called radians.

**Definition 1.** One **radian** is the angle that gives an arc length equal to the radius.

Since the circumference of a circle is  $2\pi r$ , there are  $2\pi$  radians in one revolution of the circle. However, there are also  $360^\circ$  in one revolution. This gives us a very nice formula to go back and forth:

$$2\pi \text{ rad} = 360^\circ$$

Divide both sides by 2 to get the formula

$$\pi \text{ rad} = 180^\circ$$

**Example 1.1.** Find the radian measure of  $60^\circ$  and find the degree measure of  $\frac{5\pi}{4}$

OK, so now we have a new way of indicating angles. So what? Why go through all this work? Well, using radians makes some calculations easier. For example, look at the length of an arc of a circle with angle  $\theta$

Picture goes here

Using some geometry, we can get the formula  $\frac{\theta}{2\pi} = \frac{a}{2\pi r}$  and so the length of the arc is expressed as  $a = r\theta$ .

**Example 1.2.** If the radius of a circle is 5cm, what angle gives an arc of length 6cm?

**Example 1.3.** If a circle has radius 3cm, what is the length of the arc given by the angle  $\frac{3\pi}{8}$ ?

Now, to draw an angle, we use something called the standard position. This means that we start on the positive  $x$  axis and rotate counterclockwise if  $\theta > 0$  and clockwise if  $\theta < 0$ .

More pictures. Woohoo

Note: Some angles may look like they have the exact same standard position. Just remember that  $2\pi$  sends you around one revolution. So  $\theta + 2\pi$  will look just like  $\theta$

## 1.2 Trig functions

First, remember SOCAHTOA! However, this only works for acute angles ( $\theta < \frac{\pi}{2}$ ), so what about larger angles? Draw them in standard position and use the point  $P(x, y)$  to get the ratios. Notice that this just gives you another right triangle. Here are the basic trig values that you should know:

The sign of the trig ratio can be remembered with the phrase “All Students Take Calculus”

**Example 1.4.** Compute all the trigonometric ratios for  $\theta = \frac{2\pi}{3}$

## 1.3 Trig identities

Trig functions are some of the “nicest” functions you’ll run across. Besides the fact that they’re periodic, they have a lot of other nice properties or identities. Here are the basic ones that follow directly from definitions:

csc, sec, cot, tan, and cot

Now, the next identity is probably the most useful trig identity there is. Remember the Pythagorean theorem,  $a^2 + b^2 = c^2$  and that  $\sin \theta = \frac{a}{c}$  and  $\cos \theta = \frac{b}{c}$  (draw triangle). Let’s use this to compute  $\sin^2 \theta + \cos^2 \theta$

$$\sin^2 \theta + \cos^2 \theta = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$$

So  $\sin^2 \theta + \cos^2 \theta = 1$ . The important thing here is that this works for any angle, but it has to be the same angle. We can actually use this formula to get other ones. If we divide both sides by  $\cos^2 \theta$ , we get

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$
$$\tan^2 \theta + 1 = \sec^2 \theta$$

Which is also useful. You can also divide by sin to get:

By looking at angles in the standard position, you can see that

$$\sin(-\theta) = -\sin \theta$$
$$\cos(-\theta) = \cos \theta$$

Here are some other useful identities. I won't prove these, however.

$$\begin{aligned}\sin(x + y) &= \sin x \cos y + \cos x \sin y \\ \cos(x + y) &= \cos x \cos y - \sin x \sin y\end{aligned}$$

Using the symmetry from above, we get

$$\begin{aligned}\sin(x - y) &= \sin x \cos(-y) + \cos x \sin(-y) \\ &= \sin x \cos y - \cos x \sin y \\ \cos(x - y) &= \cos x \cos(-y) - \sin x \sin(-y) \\ &= \cos x \cos y + \sin x \sin y\end{aligned}$$

There are a whole bunch of other identities, but these are the only ones we'll need.

## 1.4 Graphs of trigonometric functions

Graph  $\sin$ ,  $\cos$ , and  $\tan$ . The rest aren't very useful.

# 2 Transforming and Combining functions

## 2.1 Transformations

One way of getting a function is by starting with a function we already know and then moving it around and stretching or squashing it. The proper term for "moving it around" is called a translation. Translating means shifting up, down, left, or right without changing the actual shape of the graph. Here's how to translate a function:

First, assume that  $c > 0$ . Then,

$$\begin{aligned}f(x) + c &\text{ shifts the graph of } f(x) \text{ up by } c. \\ f(x) - c &\text{ shifts the graph of } f(x) \text{ down by } c. \\ f(x - c) &\text{ shifts the graph of } f(x) \text{ to the } \mathbf{right} \text{ by } c. \\ f(x + c) &\text{ shifts the graph of } f(x) \text{ to the } \mathbf{left} \text{ by } c.\end{aligned}$$

The term for "stretching" and "squashing" is actually just stretching. This how they work:

First, assume that  $c > 1$ . Then,

$$\begin{aligned}cf(x) &\text{ stretches the graph of } f(x) \text{ vertically by } c. \\ \frac{1}{c}f(x) &\text{ compresses the graph of } f(x) \text{ vertically by } c. \\ f(cx) &\text{ compresses the graph of } f(x) \text{ horizontally by } c. \\ f\left(\frac{x}{c}\right) &\text{ stretches the graph of } f(x) \text{ horizontally by } c.\end{aligned}$$

One last option is to reflect the graph. This just means that you flip it across one of the axes.

$-f(x)$  reflects  $f(x)$  across the  $x$ -axis.  
 $f(-x)$  reflects  $f(x)$  across the  $y$ -axis.

**Example 2.1.**  $\cos x$ ,  $\sqrt{x}$ , and  $x^2$

Go over period and amplitude as well.

## 2.2 Combining Functions

Now, instead of just playing with functions like they were play-doh, we can also combine functions that we already have. First, though, some notation.

Let  $A$  and  $B$  be sets.  $A \cap B$  is the set of elements that are in both  $A$  and  $B$ .  $A \cup B$  is the set of elements that are in either  $A$ ,  $B$ , or both. One way we combine functions is by using basic arithmetic. These operations are defined by:

Let  $f$  and  $g$  be functions and let  $A$  be the domain of  $f$  and  $B$  be the domain of  $g$ .

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \text{ and the domain is } A \cap B \\(f - g)(x) &= f(x) - g(x) \text{ and the domain is } A \cap B \\(fg)(x) &= f(x)g(x) \text{ and the domain is } A \cap B \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \text{ and the domain is } A \cap B \text{ except that } g(x) \neq 0\end{aligned}$$

**Example 2.2.** These are usually really hard to draw, but find  $f \pm g$ ,  $fg$ ,  $f/g$  for  $f(x) = \sqrt{x}$  and  $g(x) = \sqrt{4 - x^2}$

## 2.3 Composing Functions

Another method of combining functions is called composition. First, an example. Let  $f(u) = \sqrt{u}$  and let  $u = g(x) = x^2 + 1$ .  $y$  is a function of  $x$  and we compute it by

$$f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

In general, if you're given two functions  $f(x)$  and  $g(x)$ , you can compose them to get  $h(x) = f(g(x))$ . We usually denote this by using a little circle,  $\circ$ .

$$(f \circ g)(x) = f(g(x))$$

Remember  $f \circ g$  means you do  $g$  **first** and then you do  $f$ .

**Example 2.3.**  $f(x) = x^2$  and  $g(x) = x - 3$

Domains for compositions can be a little trickier than for adding or multiplying.

**Example 2.4.**  $f(x) = x^2 + 4x + 3$  and  $g(x) = \frac{1}{\sqrt{x}}$ . Work with  $f \circ g$ ,  $g \circ f$ ,  $f \circ f$ , and  $g \circ g$ .