1 Functions

1.1 What is a function?

All a function is, is something that takes a number and turns it into another number.

Example 1.1. Remember from geometry class the formula for a circle, $A = \pi r^2$. This is a nice example of a function. It takes a number, r, and spits out another number, A.

Example 1.2. For a function, we don't need a simple formula. The cost, C, of mailing a letter that weights w follows a very specific formula, but it's not as nice as the formula for the area of a circle.

Example 1.3. Functions don't even need to have formulas. Think of the population of the world for a given year. This takes a number, the year, and spits out another number, the population. There's no real formula, however. And we can't plug in future years and expect to get back an answer that's even close

Now for a better definition

Definition 1. A **function** f is a rule that assigns to each element x in a set A *exactly* one element, called $f(x)$, in a set B.

What does this mean? Look at this diagram:

The set A is called the **domain** of f and represents every value you can plug into f and get out another number. Now, a function may not hit every number in the set B. But the set of points that $f(x)$ takes on is called the **range** of f. A symbol that represents an element of the domain is called the independent variable. A symbol that represents an element in the domain is called a dependent variable.

Example 1.4. Let's go back to the area of a circle, $A = \pi r^2$. Sure, we can plug any number into r, but remember that it represents a length, so we're not going to be plugging in any negative numbers. Anything else goes, however, so the domain is all positive real numbers. Also, A is an area, so it will certainly never be negative and thus the range will be all positive numbers. Based off this formula, the area *depends* on the radius, so r is the independent variable and A is the dependent variable.

Another way to find the domain is to think of the points you can't plug in. Try to find the points where, if you were to plug them in, the function "breaks". This is usually most obvious when the function is represented as a formula.

Example 1.5. a) $f(x) = \sqrt{x+5}$, b)$g(x) = \frac{1}{x^2-x}$

I say it's more obvious because you can look at the function and you can usually tell the basic idea of where it will break. You can use this to find nice formulas where it breaks. So what about those two?

a) We know that we can only take the square root of a positive number. So whatever is under the square root symbol has to be bigger than or equal to 0. We write this as $x + 5 \geq 0$. We can solve this inequality to get $x \geq -5$. We write this as either $x \geq -5$ or using the interval notation $[-5, \infty)$

b) Well, with fractions, we know that we can't divide by zero but anything else is just fine. So we're only worried about where the denominator is 0. We write this as $x^2 - x = 0$, or $x(x-1) = 0$. This is easy to solve and find that $x = 0$ or $x = 1$. We can write this as either $\{x|x \neq 0, x \neq 1\}$, all real numbers except 0 and 1, or $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

1.2 How can we write functions?

Glad you asked. There are four ways to write, or represent, a function.

- Verbally - Use words
- Numerically - Use a table of values
- Visually - Use a graph
- Algebraically - Use a formula

Example 1.6. Algebraic: Again, we use the area of a circle. $A = \pi r^2$ is an explicit formula

Example 1.7. Numerically: The population of the world we get from a table.

Example 1.8. Verbally: Weight of a letter. The current cost to send a letter is \$0.39 for up to one ounce and then \$0.24 per ounce after that up to 13 ounces.

Example 1.9. Visually:

Example 1.10. Think of the population of the world. Again.

t	$P(t)$
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

We could try some nice method to get a formula for and get a graph, or you could just draw a curve that "looks right". This changes a numeric representation into a visual one. This is very useful for finding trends in data.

Example 1.11. A rectangular storage container with a closed top has a volume of 20m^3. The length of its base is twice the width. Material for the base and top costs \$10 per square meter and material for the sides costs \$6 per square meter. Express the costs as a function of the width and the base. This takes a verbal description and turns it into an algebraic one.

Another important thing to remember is that in the definition of a function, I used the phrase "exactly one". This is important. If something assigns more than one value to a number, then it's not a function. It's just a thingamajig and we usually don't care about thingamajigs in this class. So when is something a function? It's usually pretty obvious with most representations. Algebraic and numerical representations are almost always functions. Visual and verbal representations are the tricky ones. For verbal, the best way is to turn it into one of the other representations and try to see. For visual, we have the vertical line test.

Vertical Line Test. *A curve in the xy-plane is the graph of a function if and only if no vertical line intersects the curve more than once.*

Example 1.12. Look at $x = y^2 - 2$. This isn't a function of x because it fails the vertical line test. However, it is a function of y.

1.3 Piecewise Functions

Some functions can be described in formulas, but the formula depends on where in the domain it is. These type of functions are called **piecewise**.

Example 1.13. Define f as

$$f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$$

Find $f(0), f(1)$, and $f(2)$ and sketch the graph.

There's no limit on how many different formulas you can put into a piecewise function. For instance, you can define the cost of mailing an envelope as a piecewise function with 13 formulas. A very important example of a piecewise function is the absolute value function.

Definition 2. The **absolute value** of x is the distance from x to 0 on the number line. Distances in math are distances in real life, so they're always positive. The formula for this is:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Example 1.14. Sketch the graph of $|x|$

1.4 Symmetry

An important aspect of functions is what's called **symmetry**. A symmetric picture is one that looks the same on both halves and the same goes for functions. The right half of a function (the positive numbers) looks like the left half (the negative numbers). It may be flipped, but it still looks the same.

One form of symmetry is an **even function**. This is a function where $f(-x) = f(x)$. As an example, $f(x) = x^2$ is even. $f(-x) = (-x)^2 = (-1)^2x^2 = x^2 = f(x)$. What does this mean? When plotting the graph, you only need to plot the positive (or negative, whichever is easier) and then just reflect it over the y-axis.

Another form of symmetry is an **odd function**. This a function where $f(-x) = -f(x)$, like $f(x) = x$. With these kind of functions, you plot one half then reflect and flip.

Example 1.15. Determine if the following functions are even, odd, or neither.
a) $f(x) = x^7 + x^3$ b) $g(x) = 1 - x^4$ c) $h(x) = 2x - x^4$
a) $f(-x) = (-x)^7 + (-x)^3 = (-1)^7x^7 + (-1)^3x^3 = -x^7 - x^3 = -f(x)$
b) $g(-x) = 1 - (-x)^4 = 1 - (-1)^4x^4 = 1 - x^4 = g(x)$
c) $h(-x) = 2(-x) - (-x)^4 = -2x + (-1)^4x^4 = -2x + x^4$

1.5 Increasing and Decreasing functions

Definition 3. A function is **decreasing** if $f(x) > f(y)$ whenever $x > y$ and **increasing** if $f(x) < f(y)$ whenever $x > y$

These are very important properties, but we'll look at them more fully in another chapter.

2 The basic types of functions

Even though there are an infinite number of functions out there, we can still classify many of them. By classifying them, we learn about properties that they share. This is very important for modeling. A model is a mathematical description of something in the real world. We'll probably never have an exact formula for most things, but we can find out information about their general behavior.

2.1 Linear

A **linear** function of x is a function that has a straight line for the graph. You probably remember the slope-intercept form $y = f(x) = mx + b$, where m is the slope and b is the y-intercept. Another form that comes in handy is the point-slope form, $y - y_0 = m(x - x_0)$ or $y = m(x - x_0) + y_0$, where (x_0, y_0) is a point on the line and m is the slope.

Example 2.1. As dry air moves upward, it expands and cools. If the ground temperature is 20° C and the temperature at 1km is 10°C, express the temperature T as a function of the height, h, assuming that a linear model is appropriate. What does the slope represent? What does the model say the temperature will be at 1.5 km? At what level will the temperature be below 0°C.

2.2 Polynomials

A function P is a polynomial if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $n \geq 0$ and all the a_i are constants. Keep in mind that any of them might be 0. Polynomials have a very nice property - their domain is all real numbers, sometimes written as $\mathbb{R}$ or $(-\infty, \infty)$. If $a_n \neq 0$, we say that the **degree** of the polynomial is n.

Example 2.2. $P(x) = 3x^8 + \sqrt{(7)}x^4 - \pi x^2 + x - \frac{1}{2}$ is a polynomial of degree 8.

A polynomial of degree one is? Ans: a linear function.

A polynomial of degree 2 is often written as $P(x) = ax^2 + bx + c$ and is called a quadratic function. The graph of a quadratic is parabola and will open up if $a > 0$ and open down if $a < 0$ (draw examples)

A polynomial of degree 3 is written as $P(x) = ax^3 + bx^2 + cx + d$ and the graph also has a distinct shape.

2.3 Power Functions

A power function is a function that has the form $f(x) = x^a$, where a can be any number. This might seem like a special case of the polynomials, but there's more here.

If a is a positive integer (1,2,3,4,...), then it is a polynomial. Here are some graphs of common ones:

If $a = \frac{1}{n}$, where n is apositive integer, then $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is called a root function. *Show what happens when n is even and odd.

If $a = -1$, we have a reciprocal function, $f(x) = x^{-1} = \frac{1}{x}$. This looks like:

2.4 Rational Functions

A rational function is the ratio or quotient of two polynomials, $f(x) = \frac{P(x)}{Q(x)}$. The domain is everywhere except for where $Q(x) = 0$. There's not much we can say in general, because f will vary wildly depending on what P and Q are.

Example 2.3. $f(x) = \frac{3x^3-x+1}{x^2+2x+1}$ is a rational function defined everywhere except for $x = -1$

2.5 Algebraic Functions

A function f is algebraic if you can construct it from your basic algebraic operations: arithmetic and taking roots.

Example 2.4. $f(x) = \sqrt{(}1 - x^2)$ and $g(x) = \frac{\sqrt[3]{x^4-1}}{x-\sqrt{x}}$ are both algebraic

2.6 Trig Functions

These are incredibly important and we will be reviewing them. These are your basic sin, cos, and tan as well as csc, sec, and cot. Also keep in mind that in calculus, we always use radians and not degrees. So $f(x) = \sin x$ means the sine of the angle whose radian measure is x. So instead of going from 0 to 360, we go from 0 to 2π. The table at the beginning of the book is very useful for the important angles. These are the graphs of sin and cos:

The domain for sin and cos is $\mathbb{R}$ or $(-\infty, \infty)$ and the range is $[-1, 1]$ They are both also periodic, which just means that they repeat. In fact, $\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$. This makes them good for modelling things that repeat.

Tangent is a different beast. Recall that $\tan x = \frac{\sin x}{\cos x}$. This is defined everywhere except when $\cos x = 0$. Here's the graph:

I'll go more in depth when we get to Appendix D.

2.7 Exponential

Exponential functions are of the form $f(x) = a^x$, where a is a positive number called the base. Exponentials have domain of $\mathbb{R}$ and range of $(0, \infty)$. Here are 2 examples:

2.8 Logarithmic

A logarithmic function is of the form $f(x) = \log_a x$, where a is a positive number called the base. $\log_a x$ is the number that you have to raise a to in order to get x. Logs are the inverses of exponentials. The domain is $(0, \infty)$ and the range is $\mathbb{R}$

Both exponentials and logs will be studied more in chapter 7.

2.9 Transcendental

Transcendental functions are any functions that aren't algebraic. This includes trig functions, exponentials, and logs.

Example 2.5. Classify $f(x) = 5^x$, $g(x) = x^5$, $h(x) = \frac{1+x}{1-\sqrt{x}}$, and $u(t) = 1 - t + 5t^4 - \sin x$