Homework # 6
Due: Never

1 New Material

1. Use the guidelines of this section to sketch the curve.
   (a) $f(x) = 20x^3 - 3x^4$
      i. Domain
      f is a polynomial, so its domain is all real numbers.
      ii. Intercepts
      The y-intercept is $f(0) = 0$.
      The x-intercepts are where $f(x) = 0$, so $20x^3 - 3x^4 = 0$. Factor out an x^3 and get $x^3(20 - 3x) = 0$, so $x = 0, 20/3$
      iii. Symmetry
      There are no trig functions, so it's not periodic. $f(-x) = 20(-x)^3 - 3(-x)^4 = -20x^3 - 3x^4$ which is neither $f(x)$ nor $-f(x)$, so f is neither even nor odd.
      iv. Asymptotes.
      It's not undefined anywhere, so there won't be any vertical asymptotes.

$$\begin{aligned}\lim_{x\to\infty} 20x^3 - 3x^4 &= \lim_{x\to\infty} x^3(20 - 3x)\\ &= -\infty\end{aligned}$$

      Using the same method, you can see that $\lim_{x\to-\infty} f(x) = \infty$ and so there are no horizontal asymptotes.
      v. Intervals of Increase and Decrease
      $f'(x) = 60x^2 - 12x^3$. To look for intervals of increase and decrease, we first find the critical numbers. $f'(x)$ is a polynomial, so it's defined everywhere. The only critical numbers are where $f'(x) = 0$. $60x^2 - 12x^3 = 12x^2(5 - x)$, so $f'(x) = 0$ when $x = 0, 5$. This gives us 3 intervals:

x	$f'(x)$
$x < 0$	$+$
$0 < x < 5$	$+$
$5 < x$	$-$

      Increasing: $(-\infty, 0)$, $(0, 5)$
      Decreasing: $(5, \infty)$
      vi. Minima and Maxima
      At 0, the function doesn't change direction, so it's neither a local max nor a min. At 5, the function changes from increasing to decreasing, so it's a maximum.
      Local Maximum: $x = 5$
      Local Minimum: None

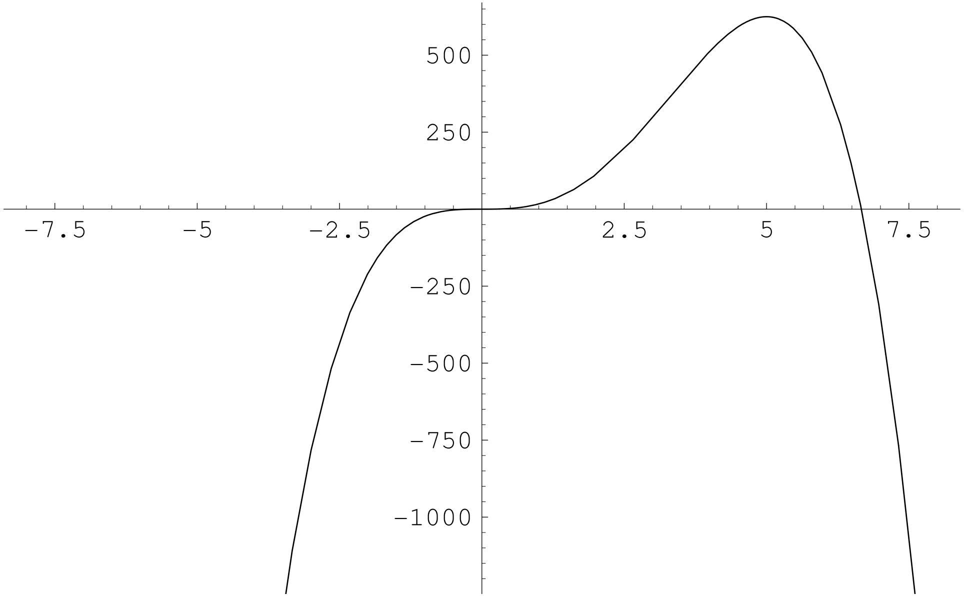
vii. Intervals of concavity and inflection points.

$f''(x) = 120x - 36x^2$. Inflection points can be where this is zero or undefined. It's never undefined, so we're only looking for $f''(x) = 0$. $120x - 36x^2 = 0$ so $12x(10 - 3x) = 0$. Thus possible inflection points are $x = 0, 10/3$ and we have 3 intervals to check.

x	$f''(x)$
$x < 0$	$-$
$0 < x < 10/3$	$+$
$10/3 < x$	$-$

CU: $(0, 10/3)$
CD: $(-\infty, 0)$, $(10/3, \infty)$
Inflection points at $x = 0, 10/3$ since the concavity changes at both points.

viii. Graph



(b) $f(x) = \frac{x^2}{x^2-9}$

i. Domain

It's a rational function, so its domain is everywhere except where the denominator is 0. Thus the domain is all numbers except $x = 3, -3$.

ii. Intercepts

y-intercept: $f(0) = \frac{0}{-9} = 0$, so the y-intercept is 0.

x-intercept: $\frac{x^2}{x^2-9} = 0$, so $x = 0$ is the x-intercept.

iii. Symmetry

$f(-x) = \frac{(-x)^2}{(-x)^2-9} = \frac{x^2}{x^2-9} = f(x)$ so the function is even.

iv. Asymptotes

The function is undefined at $x = 3, -3$, but we're only interested in the positive x values since f is even. Let's look at one of the one-sided limits at $x = 3$.

$$\begin{aligned}\lim_{x\to 3^+}\frac{x^2}{x^2-9} &= \lim_{x\to 3^+}\frac{1}{1-9/x^2}\\ &= \infty\end{aligned}$$

It's ∞ since the denominator goes to zero and the numerator goes to 1, so the whole thing is an infinite limit. It's positive because when $x > 3$, $1-9/x^2 > 0$. The left hand limit gives you $-\infty$, which will be useful for graphing.

$$\begin{aligned}\lim_{x\to \infty}\frac{x^2}{x^2-9} &= \lim_{x\to \infty}\frac{1}{1-9/x^2}\\ &= 1\end{aligned}$$

So,
VA: $x = 3, -3$ (the one at -3 follows from symmetry)
HA: $y = 1$

v. Intervals of Increase and Decrease
First, $f'(x) = \frac{-18x}{(x^2-9)^2}$. It's undefined at $x = 3, -3$, so those are critical values. It's zero at $x = 0$, which is another critical value.

x	$f'(x)$
$x < -3$	$+$
$-3 < x < 0$	$+$
$0 < x < 3$	$-$
$3 < x$	$-$

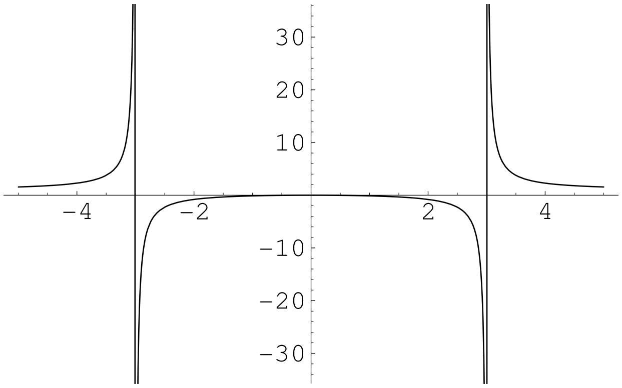
vi. Mins and maxes
At zero, the graph changes from increasing to decreasing, so it's a local max. f' doesn't change sign at the other critical values, so they're neither mins nor maxes.
Local Max: $x = 0$
Local Min: None

vii. Intervals of Concavity and Inflection Points
$f''(x) = \frac{54(3+x^2)}{(x^2-9)^3}$, so it's undefined at $x = 3, -3$ and is never zero.

x	$f''(x)$
$x < -3$	$+$
$-3 < x < 3$	$-$
$3 < x$	$+$

CU: $(-\infty, -3)$,$(3, \infty)$
CD: $(-3, 3)$
$x = 3, -3$ are both inflection points.

viii. Graph



(c) $f(x) = \sin x$

This one was done in class

2. If 1200 cm^2 is available to make a box with a square base and an open top, find the largest possible volume of the box. (Hint: You'll need to consider the area of each side. But remember which sides you have and which you don't.)

   Let the base of the box have dimensions x, y and have height z. Since it has a square base, the base has dimensions x, x. So the volume of the box is $V = x^2 z$. We want to get rid of the z, so we use the fact that we have 1200 cm^2 of material.

$$x^2 + 4xz = 1200$$

   Solve this for z to get $z = \frac{1200-x^2}{x}$. Plug this into the equation for volume to get

$$\begin{aligned} V &= x^2\left(\frac{1200 - x^2}{x}\right) \\ &= x(1200 - x^2) \\ &= 1200x - x^3 \end{aligned}$$

   To find the maximum, we take the derivative and set it equal to zero.

$$\frac{dV}{dx} = 1200 - 3x^2 = 0$$

   Thus $x = \pm 20$. However, we won't have a box with a negative dimension, so the only critical number we care about is $x = 20$. We have to test this to make sure it's a maximum, however. We'll use the second derivative test here.

$$\frac{d^2V}{dx^2} = -6x$$

   Evaluate this at $x = 20$ to get -120, which is negative. Thus $x = 20$ and $z = 40$ is a maximum by the second derivative test. The maximum volume is $20 * 20 * 40 = 16000$

2 Review Material

This section has problems that might be like the ones on the exam. They won't be exactly like this, and it's possible that not all the material here will be covered on the exam. Basically, if it's on here then it's fair game for the exam. If it's not on here, then still learn it, but don't worry about it for the exam. Keep in mind two things, however. First, just because the material isn't given an explicit problem here, it still may be needed to do another problem. Second, all trig functions are valid for all of the material (limits, derivatives, graphing, etc.) and step functions are valid for problems involving limits.

1. Graph the following functions not by plotting points or using the material covered at the end of the course, but by starting with a known function and transforming it.
   (a) $y = -\sin x$
   (b) $y = \frac{1}{x+2}$
   (c) $y = 1 + \frac{1}{2}x^3$
2. If $f(x) = \sqrt{x}$ and $g(x) = \sin x$, then find $f \circ g$ and $g \circ f$ as well as their domains.
   (a) $(f \circ g)(x) = \sqrt{\sin x}$
   The domain of $\sin x$ is all real numbers, so anything can go into $g(x)$. However, the domain of $f(x)$ is only the nonnegative numbers. This means that the only numbers that can go into $f \circ g$ are the ones where $\sin x > 0$. $\sin x$ is positive on $[0, \pi]$, $[2\pi, 3\pi]$, ... It's also positive on $[-2\pi, -\pi]$, $[-4\pi, -3\pi]$, ... Therefore, that is the domain.
   (b) $(g \circ f)(x) = \sin(\sqrt{x})$
   The domain of $\sqrt{x}$ is the positive numbers. So what we want to look it is the positive numbers x such that $\sqrt{x}$ is in the domain of f. However, the domain of f is all real numbers, so the domain of $g \circ f$ is all positive numbers or $[0, \infty)$
3. Evaluate $\lim_{x \to -2} \sqrt{x^4 + 3x + 6}$ using the limit laws. You don't need to state the exact limit law you're using.

$$\begin{aligned}
\lim_{x \to -2} \sqrt{x^4 + 3x + 6} &= \sqrt{\lim_{x \to -2} (x^4 + 3x + 6)} \\
&= \sqrt{\lim_{x \to -2} x^4 + \lim_{x \to -2} 3x + \lim_{x \to -2} 6} \\
&= \sqrt{\lim_{x \to -2} x^4 + \lim_{x \to -2} 3x + 6} \\
&= \sqrt{\lim_{x \to -2} x^4 + 3 \lim_{x \to -2} x + 6} \\
&= \sqrt{(-2)^4 + 3(-2) + 6} \\
&= \sqrt{16 - 6 + 6} \\
&= 4
\end{aligned}$$

4. Find the following limits.

(a) $\lim\limits_{x\to 3} \frac{x^2-9}{x^2+2x-3}$

$$\begin{aligned} \lim_{x\to 3} \frac{x^2-9}{x^2+2x-3} &= \frac{9-9}{9+6-3} \\ &= 0 \end{aligned}$$

(b) $\lim\limits_{x\to 0} \cos(x+\sin x)$

$$\begin{aligned} \lim_{x\to 0} \cos(x+\sin x) &= \cos\left(\lim_{x\to 0}(x+\sin x)\right) \\ &= \cos(0+\sin 0) \\ &= \cos(0) \\ &= 1 \end{aligned}$$

(c) $\lim\limits_{x\to 4^+} \frac{4-x}{|4-x|}$

For the right-sided limit, $x > 4$, so $4-x<0$ and thus $|4-x| = -(4-x)$. This means

$$\begin{aligned} \lim_{x\to 4^+} \frac{4-x}{|4-x|} &= \lim_{x\to 4^+} \frac{4-x}{-(4-x)} \\ &= \lim_{x\to 4^+} (-1) \\ &= -1 \end{aligned}$$

(d) $\lim\limits_{x\to 0} \frac{1-\sqrt{1-x^2}}{x}$

$$\begin{aligned} \lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x} &= \lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x}\left(\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right) \\ &= \lim_{x\to 0} \frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})} \\ &= \lim_{x\to 0} \frac{x^2}{x(1+\sqrt{1-x^2})} \\ &= \lim_{x\to 0} \frac{x}{1+\sqrt{1-x^2}} \\ &= \frac{0}{1+\sqrt{1-0}} \\ &= 0 \end{aligned}$$

(e) $\lim_{x\to\infty} \frac{3x+5}{x-4}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x+6}{x-4} &= \lim_{x\to\infty} \frac{3x+6}{x-4}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) \\
&= \lim_{x\to\infty} \frac{3+\frac{6}{x}}{1-\frac{4}{x}} \\
&= \frac{3}{1} \\
&= 1
\end{aligned}$$

(f) $\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} &= \lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) \\
&= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x^2}}} \\
&= \frac{1}{\sqrt{9}} \\
&= \frac{1}{3}
\end{aligned}$$

(g) $\lim_{x\to-\infty} \frac{x+x^3+x^5}{1-x^2+x^4}$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x+x^3+x^5}{1-x^2+x^4} &= \lim_{x\to-\infty} \frac{x+x^3+x^5}{1-x^2+x^4}\left(\frac{\frac{1}{x^5}}{\frac{1}{x^5}}\right) \\
&= \lim_{x\to-\infty} \frac{\frac{1}{x^4}+\frac{1}{x^2}+1}{\frac{1}{x^5}-\frac{1}{x^3}+\frac{1}{x}} \\
&= \infty
\end{aligned}$$

To see that this is ∞ and not $-\infty$ actually needs a technique I forgot to go over in class. The rest of the problem should be doable.

5. Find y' for the following:

(a) $y = \cos(\tan x)$

$$\begin{aligned}
\frac{dy}{dx} &= -\sin(\tan x)\frac{d}{dx}(\tan x) \\
&= -\sin(\tan x)(\sec^2 x)
\end{aligned}$$

(b) $y = \frac{3x-x}{\sqrt{2x+1}}$

First, I meant for this to be $3x - x^2$ in the numerator, so that's how I'll do the problem.

Personally, I'd first rewrite this to get $y = (3x - x^2)(2x + 1)^{-1/2}$

$$\frac{dy}{dx} = (3x - x^2)(-1/2)(2x + 1)^{-3/2}(2) + (2x + 1)^{-1/2}(3 - 2x)$$

(c) $y = \frac{1}{\left(x+\frac{1}{x^2}\right)^3}$

Again, I'd rewrite this so that $y = \left(x + \frac{1}{x^2}\right)^{-3}$

$$\frac{dy}{dx} = (-3)\left(x + \frac{1}{x^2}\right)^{-4}\left(1 - \frac{2}{x^3}\right)$$

(d) $y = \frac{\tan\sqrt{x}}{1+\sin x}$ (This one will be messy)

Again, there's no real need to use the quotient rule. $y = (\tan(\sqrt{x}))\,(1 + \sin x)^{-1}$

$$\frac{dy}{dx} = \tan(\sqrt{x})(-1)\,(1 + \sin x)^{-2}\,(\cos x) + (1 + \sin x)^{-1}\,(\sec^2\sqrt{x})(1/2)(x^{-1/2})$$

(e) $xy^4 + x^2y = x + 3y$

$$\begin{aligned} y^4 + x(4y^3)(y') + 2xy + x^2y' &= 1 + 3y' \\ 4xy^3y' + x^2y' - 3y' &= 1 - y^4 - 2xy \\ y'\left(4xy^3 + x^2 - 3\right) &= 1 - y^4 - 2xy \\ y' &= \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3} \end{aligned}$$

(f) $\sin(xy) = x^2 - y$

$$\begin{aligned} \cos(xy)\,(y + xy') &= 2x - y' \\ y\cos(xy) + xy'\cos(xy) &= 2x - y' \\ xy'\cos(xy) + y' &= 2x - y\cos(xy) \\ y'\,(x\cos(xy) + 1) &= 2x - y\cos(xy) \\ y' &= \frac{2x - y\cos(xy)}{x\cos(xy) + 1} \end{aligned}$$

6. Find an equation of the tangent line to the curve $y = 4\sin^2 x$ at the point $(\pi/6, 1)$

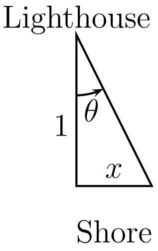
First, $\frac{dy}{dx} = 8\sin x \cos x$ (using the chain rule). For a line, we need a point and a slope. The point is $(\pi/6, 1)$, so we just plug in $\pi/6$ into $\frac{dy}{dx}$ to get the slope.

$\frac{dy}{dx}|_{x=\pi/6} = 8\sin(\pi/6)\cos(\pi/6) = 8(1/2)(\sqrt{3}/2) = 2\sqrt{3}$

So the line is $y - 1 = 2\sqrt{3}(x - \pi/6)$

IMPORTANT: Your answer should be able to be written as $y = mx + b$. If there's anything more complicated than an x multiplied by a number and a y multiplied by a number, such as a trig function or an x^n or something, you don't have a line.

7. A lighthouse is located on a small island 3km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute.. How fast is the beam of light moving along the shoreline when it is 1km away from P?



We know that the lighthouse makes four revolutions per minute, which means it's revolving at 8π radians per minute.

Know: $\frac{d\theta}{dt} = 8\pi$

Want: $\frac{dx}{dt}$ when $x = 1$

We can relate these with the equation $\tan\theta = x$ Now we use implicit differentiation:

$$\begin{aligned} \tan\theta &= x \\ \sec^2\theta\frac{d\theta}{dt} &= \frac{dx}{dt} \\ 8\pi\sec^2\theta &= \frac{dx}{dt} \end{aligned}$$

So if we can find θ when $x = 1$, we're all set. θ, however, won't be a nice angle. Fortunately, we don't actually need θ itself. All we care about is $\sec\theta$, which we can find. If $x = 1$, then the hypotenuse of the triangle is $\sqrt{1+3^2} = \sqrt{10}$. $\sec\theta = \frac{1}{\cos\theta}$, so $\sec^2\theta = (\sqrt{10}/3)^2 = 10/9$

Thus $\frac{dx}{dt} = (10/9)(8\pi) = \frac{80\pi}{9}$ when $x = 1$.

8. Find the local maximum and minimum values of $f(x) = x^3 - 27x + 10$ using both the first and second derivative tests.

For maxes and mins, we'll need the first derivative, so $f'(x) = 3x^2 - 27$. It's defined everywhere, so the only critical values are where it's zero.

$$\begin{aligned} 3x^2 - 27 &= 0 \\ 3x^2 &= 27 \\ x^2 &= 9 \\ x &= 3, -3 \end{aligned}$$

For the first derivative test, we check the following intervals

x	$f'(x)$
$x < -3$	$+$
$-3 < x < 3$	$-$
$3 < x$	$+$

At $x = -3$, the first derivative goes from positive to negative, so $x = -3$ is a max.

At $x = 3$, the first derivative goes from negative to positive, so $x = 3$ is a min.

For the second derivative test, we need the second derivative. $f''(x) = 6x$.

$f''(-3) = -18$, which is negative. By the second derivative test, $x = -3$ is a max.

$f''(3) = 18$, which is positive. By the second derivative test, $x = 3$ is a min.

9. Find both the local and absolute maxima and minima of the function on the given interval.

   (a) $f(x) = x - \sqrt{x}$, [0,4]
   $f'(x) = 1 - \frac{1}{2\sqrt{x}}$
   It's undefined when $x \leq 0$ and it's zero when $x = \frac{1}{4}$. However, the only points that are in the interval are $x = 0, \frac{1}{4}$.

x	$f(x)$
0	0
$\frac{1}{4}$	$\frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$
4	2

   Thus $x = \frac{1}{4}$ is an absolute minimum and $x = 4$ is an absolute maximum.

   (b) $f(x) = \frac{x}{x^2+x+1}$, $[-2, 0]$
   $f'(x) = \frac{1-x^2}{(x^2+x+1)^2}$
   The denominator is never 0, so we only have to worry about where $f'(x) = 0$, which is $x = 1, -1$. The only one that is in the interval is $x = -1$

x	$f(x)$
-2	$-2/3$
-1	-1
0	0

Thus $x = -1$ is an absolute minimum and $x = 0$ is an absolute maximum.

(c) $f(x) = \sin x + \cos^2 x,\ [0, \pi]$
$f'(x) = \cos x - 2 \sin x \cos x$
$f'(x)$ is never undefined, so we look for $f'(x) = 0$

$$\begin{aligned} \cos x - 2 \sin x \cos x &= 0 \\ \cos x \,(1 - 2 \sin x) &= 0 \end{aligned}$$

So either $\cos x = 0$, which means $x = \pm\pi/2, \pm 3\pi/2, \ldots$, or $\sin x = 1/2$, which means $x = \pi/6, 5\pi/6, \ldots$ as well as infinitely many more (remember that sin and cos are periodic). However, we only care about the interval $[0, \pi]$, so the only values we care about are $x = \pi/6, \pi/2, 5\pi/6$

x	$f(x)$
0	1
$\pi/6$	$5/4$
$\pi/2$	1
$5\pi/6$	$5/4$
π	1

So f has absolute maxes at $x = 0, \pi/2, \pi$ and absolute mins at $x = \pi/6, 5\pi/6$.