Homework # 4
Due: 6/13/06

1. Differentiate the following:

   (a) $f(x) = x^2(\cos x)(\sin x)$ This is just the product rule used multiple times.

$$f'(x) = x^2 \frac{d}{dx}((\cos x)(\sin x)) + 2x(\cos x)(\sin x)$$
$$= x^2\left(\cos^2 x - \sin^2 x\right) + 2x(\cos x)(\sin x)$$

   (b) $f(x) = \frac{\tan x - 1}{\sec x}$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x \tan x)}{\sec^2 x}$$

   (c) $f(\theta) = \frac{\sin\theta(\theta + \tan\theta)}{1 + \sec\theta}$

$$f'(\theta) = \frac{(1+\sec\theta)\frac{d}{dx}(\sin\theta(1+\tan\theta)) - \sin\theta(1+\tan\theta)(\sec\theta\tan\theta)}{(1+\sec\theta)^2}$$
$$= \frac{(1+\sec\theta)\frac{d}{dx}\left(\sin\theta(\sec^2\theta) + \cos\theta(1+\tan\theta)\right) - \sin\theta(1+\tan\theta)(\sec\theta\tan\theta)}{(1+\sec\theta)^2}$$

   (d) $f(x) = \frac{(x-1)^4}{(x^2+2x)^5}$
   To make things somewhat easier, I'm going to rewrite this as $f(x) = (x-1)^4(x^2+2x)^{-5}$.

$$f'(x) = (x-1)^4(-5)(x^2+2x)^{-6}(2x+2) + (x^2+2x)^{-5}(4)(x-1)^3$$

   (e) $f(x) = \sin\left(\tan\left(\sqrt{\sin x}\right)\right)$
   This is the chain rule used multiple times

$$f'(x) = \cos\left(\tan\left((\sin x)^{1/2}\right)\right)\frac{d}{dx}\left(\tan\left((\sin x)^{1/2}\right)\right)$$
$$= \cos\left(\tan\left((\sin x)^{1/2}\right)\right)\sec^2\left((\sin x)^{1/2}\right)\frac{d}{dx}\left((\sin x)^{1/2}\right)$$
$$= \cos\left(\tan\left((\sin x)^{1/2}\right)\right)\sec^2\left((\sin x)^{1/2}\right)(\frac{1}{2})(\sin x)^{-1/2}(\cos x)$$

2. Find dy/dx by implicit differentiation:
   For all of these problems, it's incredibly important to remember the chain rule. Think of the DeLorean. You wouldn't attempt time travel without the Flux Capacitor, would you?

   (a) $1 + x = \sin(xy^2)$

$$1 = \cos(xy^2)\frac{d}{dx}(xy^2)$$
$$= \cos(xy^2)(y^2 + 2xyy')$$
$$= y^2\cos(xy^2) + 2xyy'\cos(xy^2)$$
$$1 - y^2\cos(xy^2) = 2xyy'\cos(xy^2)$$
$$\frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)} = y'$$

(b) $\tan(x-y) = \frac{y}{1+x^2}$

First, rewrite the right hand side as $y(1+x^2)^{-1}$

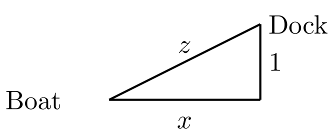
$$\begin{aligned}
\sec^2(x-y)(1-y') &= y'(1+x^2)^{-1} + y(-1)(1+x^2)^{-2}(2x) \\
\sec^2(x-y) - y'\sec^2(x-y) &= y'(1+x^2)^{-1} - 2xy(1+x^2)^{-2} \\
y'(1+x^2)^{-1} + y'\sec^2(x-y) &= \sec^2(x-y) + 2xy(1+x^2)^{-2} \\
y'\left((1+x^2)^{-1} + \sec^2(x-y)\right) &= \sec^2(x-y) + 2xy(1+x^2)^{-2} \\
y' &= \frac{\sec^2(x-y) + 2xy(1+x^2)^{-2}}{(1+x^2)^{-1} + \sec^2(x-y)}
\end{aligned}$$

3. Find the first and second derivatives of $f(x) = x^n$, where n is any number.

$$\begin{aligned}
f'(x) &= nx^{n-1} \\
f''(x) &= n(n-1)x^{n-2}
\end{aligned}$$

4. §3.9 # 16

First, we draw the picture:



We're given $\frac{dz}{dt} = 1$ and we want to find $\frac{dx}{dt}$ when $x = 8$. Now we relate all the quantities with an equation.

$$x^2 + 1 = z^2$$

We want to find $\frac{dx}{dt}$ so we use implicit differentiation and take the derivative of both sides.

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

Divide both sides by 2 to get

$$x\frac{dx}{dt} = z\frac{dz}{dt}$$

We only care about when $x = 8$. So we plug this into the original equation and get that $z = \sqrt{1+64} = \sqrt{65}$ when $x = 8$. Also using the fact that $\frac{dz}{dt} = 1$, we get the following:

$$8\frac{dx}{dt} = \sqrt{65}$$

So $\frac{dx}{dt} = \frac{8}{\sqrt{65}}$ when $x = 8$. Or the boat is moving towards the dock at a rate of $\frac{8}{\sqrt{65}}$m/s when it is 8m away.