Homework # 3
Due: 6/6/06

1. Find an equation of the tangent line to $y = \sqrt{2x+1}$ at the point (4,3).

   The equation of a tangent line to $y = f(x)$ at $(a, b)$ is $(y - b) = f'(a)(x - a)$. First, we write the equation as $y = (2x+1)^{\frac{1}{2}}$

$$\begin{aligned} \frac{d}{dx}\left((2x+1)^{\frac{1}{2}}\right) &= \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2x) \\ &= (2x+1)^{-\frac{1}{2}} \end{aligned}$$

   Evaluated at $x = 4$, we get $(2(4)+1)^{-\frac{1}{2}} = 9^{-\frac{1}{2}} = \frac{1}{3}$. So the tangent line is

$$y - 3 = \frac{1}{3}(x - 4)$$

2. Draw a graph of a function $f$ that has the properties

$$\begin{array}{ll} g(0) = 0 & g'(0) = 3 \\ g'(1) = 0 & g'(2) = 1 \end{array}$$

3. Use the limit definition of the derivative to compute $f'(a)$ if $f(x) = \frac{x^2+1}{x-2}$

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x+h)^2+1}{x+h-2} - \frac{x^2+1}{x-2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x+h)^2+1}{x+h-2}\left(\frac{x-2}{x-2}\right) - \frac{x^2+1}{x-2}\left(\frac{x+h-2}{x+h-2}\right)}{h} \\
&= \lim_{h\to 0} \frac{((x+h)^2+1)(x-2) - (x^2+1)(x+h-2)}{h(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{(x^2+2xh+h^2+1)(x-2) - (x^2+1)(x+h-2)}{h(x+h-2)(x-2)} \quad = \lim_{h\to 0} \frac{x^3+2hx^2+x^2h^2+x-2x^2-}{h(x+} \\
&= \lim_{h\to 0} \frac{hx^2+h^2x-4xh-2h^2-h}{h(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{h(x^2+hx-4x-2h-1}{h(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{x^2+hx-4x-2h-1}{(x+h-2)(x-2)} \\
&= \frac{x^2+0x-4x-2(0)-1}{(x+0-2)(x-2)} \\
&= \frac{x^2-4x-1}{(x-2)^2}
\end{aligned}
$$

So $f'(a) = \frac{a^2-4a-1}{(a-2)^2}$

4. Find $f'(x)$ and find the points where the tangent line is horizontal if $f(x) = x^2+2x+1$. Again, use the limit definition of the derivative.

   The if $f(x)$ has a horizontal tangent at $a$, then $f'(a) = 0$. So

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^2+2(x+h)+1-x^2-2x-1}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2+2x+2h+1-x^2-2x-1}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2+2h}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h+2}{h} \\
&= \lim_{h\to 0} (2x+h+2) \\
&= 2x+2
\end{aligned}
$$

So the horizontal tangents occur when $2x+2=0$ or at $x=-1$.

5. Find $f'(x)$ and its domain for

(a) $f(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$

First, $f(x) = \frac{x^{1/2}+1}{x^{1/2}-1}$

$$
\begin{aligned}
f'(x) &= \frac{(x^{1/2}-1)(\frac{1}{2}x^{-1/2}) - (x^{1/2}+1)(\frac{1}{2}x^{-1/2})}{(x^{1/2}-1)^2} \\
&= \frac{\frac{1}{2} - \frac{1}{2}x^{-1/2} - \frac{1}{2} - \frac{1}{2}x^{-1/2}}{(x^{1/2}-1)^2} \\
&= \frac{x^{-1/2}}{(x^{1/2}-1)^2} \\
&= \frac{1}{x^{1/2}(x^{1/2}-1)^2}
\end{aligned}
$$

This answer is just fine

For the domain, $x$ can't be negative since we're taking the square root of it. However, it also cannot be 0 or 1 since either one would make the denominator 0. Thus the domain is $(0,1),(1,\infty)$ or $0 < x < 1, 1 < x$.

(b) $f(x) = x^2 + x + x^{-1} + x^{-2}$

$$f'(x) = 2x + 1 - x^{-2} - 2x^{-3}$$

The only number that doesn't work here is 0 ( remember $x^{-2} = \frac{1}{x^2}$), so the domain is all real numbers except 0. You can also write $(-\infty,0),(0,\infty)$.

(c) $f(x) = (x-1)\sqrt{x}$

You could use the product rule here, but it's probably easier to multiply out and use the power rule. Remember that when multipling, you add exponents. $f(x) = x(x^{1/2}) - x^{1/2} = x^{3/2} - x^{1/2}$.

$$f'(x) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$$

Since we're taking square roots here, $x$ can't be negative. Also, $x$ can't be zero since it would make a denominator zero. Thus the domain is $(0,\infty)$ or $x > 0$.

6. A particle moves according to $s = f(t) = t^4 - 4t + 1$ for $t \geq 0$, where $t$ is seconds and $s$ is feet.

(a) Find the velocity at time $t$

The velocity is just the derivative of the position, so

$$v(t) = s'(t) = 4t^3 - 4$$

(b) When is the particle at rest?

The particle is at rest when the velocity (derivative) is 0. So we want to find where $4t^3 - 1 = 0$ . This occurs when $t^3 = 1$. The only real number this works for is 1, so the particle is at rest at $t = 1$

(c) When is the particle moving in the negative direction?

The particle is moving in the negative direction when the velocity is negative. We have two intervals to test - $t < 1$ and $t > 1$. For $t < 1$, plug in 0 to get $v(0) = -1$, which is negative. For $t > 0$ plug in any number bigger than 1, like 2. $v(2) = 8 - 1 = 7 > 0$. So the particle is moving in the negative direction when $0 < t < 1$. Remember that we don't deal with negative times.

(d) Find the total distance traveled during the first 8 seconds.

We can't just do $s(8) - s(0)$ because that just finds how far away the particle is from $s(0)$ after 8 seconds and won't take into account the fact that it started out traveling in the negative direction. Thus, we have to break this up into intervals and find out how far it traveled over each interval. The distance it travelled is $s(0) - s(1) + s(8) - s(1) = 4070$