Homework # 2

Due: 5/30/06

1. Use the graph to find the following limits:

$$\lim_{x\to 2^+} f(x) \quad \lim_{x\to 2^-} f(x) \quad \lim_{x\to 2} f(x)$$
$$\lim_{x\to 0} f(x) \quad \lim_{x\to -1} f(x)$$

2. Sketch the graph of a function that satisfies all of the given conditions:

$$\lim_{x\to 0^-} f(x) = 1 \qquad \lim_{x\to 0^+} f(x) = 2 \qquad f(0) = 1$$
$$\lim_{x\to 1} f(x) = 1 \qquad f(x) \text{ is undefined}$$

3. Use the limit laws to find the following limits. Justify each step.

   (a) $\lim_{x\to -3} (x^2+3)(x-4)$

$$\begin{aligned}
\lim_{x\to -3} (x^2+3)(x-4) &= \left(\lim_{x\to -3} (x^2+3)\right)\left(\lim_{x\to -3} (x-4)\right) && \text{Law 4}\\
&= \left(\lim_{x\to -3} x^2 + \lim_{x\to -3} 3\right)\left(\lim_{x\to -3} x - \lim_{x\to -3} 4\right) && \text{Laws 1 and 2}\\
&= \left(\lim_{x\to -3} x^2 + 3\right)\left(\lim_{x\to -3} x - 4\right) && \text{Law 7}\\
&= \left((-3)^2+3\right)(-3-4) && \text{Law 9}\\
&= -84
\end{aligned}$$

   (b) $\lim_{x\to 2} \frac{x-2}{x^3-8}$

   First, $\frac{x-2}{x^3-8} = \frac{1}{x^2+2x+4}$

$$\begin{aligned}
\lim_{x\to 2}\frac{1}{x^2+2x+4} &= \frac{\lim\limits_{x\to 2} 1}{\lim\limits_{x\to 2}(x^2+2x+4)} && \text{Law 5}\\
&= \frac{\lim\limits_{x\to 2} 1}{\lim\limits_{x\to 2} x^2+\lim\limits_{x\to 2} 2x+\lim\limits_{x\to 2} 4} && \text{Laws 1,2}\\
&= \frac{\lim\limits_{x\to 2} 1}{\lim\limits_{x\to 2} x^2+2\lim\limits_{x\to 2} x+\lim\limits_{x\to 2} 4} && \text{Law 3}\\
&= \frac{1}{\lim\limits_{x\to 2} x^2+2\lim\limits_{x\to 2} x+4} && \text{Law 7}\\
&= \frac{1}{2^2+2(2)+4} && \text{Law 9}\\
&= \frac{1}{12}
\end{aligned}$$

Note: I don't think I intended for the denominator to be x^3-8. I think I wanted x^3-2.

(c) $\lim\limits_{x\to 1}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^6$

$$\begin{aligned}
\lim_{x\to 1}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^6 &= \left(\lim_{x\to 1}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\right)^6 && \text{Law 10}\\
&= \left(\lim_{x\to 1}\sqrt{x}+\lim_{x\to 1}\frac{1}{\sqrt{x}}\right)^6 && \text{Law 1}\\
&= \left(\lim_{x\to 1}\sqrt{x}+\frac{\lim\limits_{x\to 1} 1}{\lim\limits_{x\to 1}\sqrt{x}}\right)^6 && \text{Law 5}\\
&= \left(\lim_{x\to 1}\sqrt{x}+\frac{1}{\lim\limits_{x\to 1}\sqrt{x}}\right)^6 && \text{Law 7}\\
&= \left(\sqrt{\lim_{x\to 1} x}+\frac{1}{\sqrt{\lim\limits_{x\to 1} x}}\right)^6 && \text{Law 10}\\
&= \left(\sqrt{1}+\frac{1}{\sqrt{1}}\right)^6 && \text{Law 9}\\
&= 64
\end{aligned}$$

4. Find the following limits. You don't need to justify each step, but remember to show work.

   (a) $\lim\limits_{x\to -3}\frac{x+3}{x^2-9}$

$$\lim_{x \to -3} \frac{x+3}{x^2-9} = \lim_{x \to -3} \frac{x+3}{(x+3)(x-3)}$$
$$= \lim_{x \to -3} 1x - 3$$
$$= \frac{1}{-3-3}$$
$$= -\frac{1}{6}$$

(b) $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} \left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \right)$$
$$= \lim_{x \to 0} \frac{\left(\sqrt{1+x}\right)^2 - 1^2}{x(\sqrt{1+x}+1)}$$
$$= \lim_{x \to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)}$$
$$= \lim_{x \to 0} \frac{x}{x(\sqrt{1+x}+1)}$$
$$= \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1}$$
$$= \frac{1}{\sqrt{1+0}+1}$$
$$= \frac{1}{2}$$

5. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} x^2 \sin(1/x) = 0$$

$$\begin{array}{ccccc} -1 & \leq & \sin\left(\frac{1}{x}\right) & \leq 1 & \\ -x^2 & \leq & x^2 \sin\left(\frac{1}{x}\right) & \leq & x^2 \end{array}$$

Since $\lim_{x \to 0} -x^2 = 0$ and $\lim_{x \to 0} x^2 = 0$, by the squeeze theorem, $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

6. Show that $f(x) = \sqrt{3}x^2 + 2$ is continuous at $x = -5$

Step 1: $f(-5) = \sqrt{3}(-5)^2 + 2 = 25\sqrt{3} + 2$

Step 2:

$$\begin{aligned}
\lim_{x\to-5} \sqrt{3}x^2 + 2 &= \lim_{x\to-5} \sqrt{3}x^2 + \lim_{x\to-5} 2 \\
&= \sqrt{3} \lim_{x\to-5} x^2 + \lim_{x\to-5} 2 \\
&= \sqrt{3} \lim_{x\to-5} x^2 + 2 \\
&= \sqrt{3}(-5)^2 + 2 \\
&= 25\sqrt{3} + 2
\end{aligned}$$

Step 3: $\lim_{x\to-5} f(x) = f(-5)$, so f is continuous at -5.

7. Find an equation of the tangent line to $y = \sqrt{2x+1}$ at the point (4,3).

   Didn't have to do this one on this homework.