Contents

Chapter 1

Exponents and Polynomials

In this chapter, we discuss...

1.1 Basics

Time to remember what you are supposed to know. How do we find the area of a rectangle? Length times Width. In fact, let's make it even easier. How do we find the area of a square? If the length of a side is ℓ, we would just say the area is ℓ^2 (or $\ell \times \ell$) which is ℓ *squared*.

The question now becomes: Why do we write ℓ^2? What is that little 2? This little 2 is called an *exponent* and exponents are all we're going to talk about for the first half of this section.

Example 1

What exponent do we use to find the volume of a cube?

First of all, let's remember what a cube is. It's a 3-dimensional object with 6 sides, all of which are squares (see Figure 1.1)

Figure 1.1: Dice

If you remember some more geometry, the volume of a rectangular solid is given by length $\times$ width $\times$ height. If all of these are the same, which they are in the case of a cube, we end up with length $\times$ length $\times$ length. If our length is ℓ, we end up with $\ell \times \ell \times \ell = \ell^3$ which is read ℓ *cubed*. Fitting, isn't it, since we're looking at a cube.

Therefore, we have established that the exponent we need is a 3.

1.1.1 Properties of Exponents

Let's look at this more generally now. The exponent is really just counting how many times we want to multiply something by itself.

Example 2

Find: (a) 2^3, (b) 2^4, (c) 1^{50}, and (d) 0^{37}.

(a) $2^3 = 2 \times 2 \times 2 = 4 \times 2 = 8$.

(b) $2^4 = 2 \times 2 \times 2 \times 2 = 4 \times 2 \times 2 = 8 \times 2 = 16$.

(c) $1^{50} = 1 \times 1 \times \cdots \times 1 = 1$ since 1 times anything is still whatever you started with.

(d) $0^{37} = 0 \times 0 \times \cdots \times 0 = 0$ since 0 times anything is always 0.

Let's focus on parts (a) and (b) for a minute. Notice that in part (b) we're multiplying by 2 one time more than in part (a). This makes sense because we end up with twice as much. The point being: For any numbers x and a,

$$x^{a+1} = x \cdot x^a.$$

In fact, we can be even more general about this and get our first property of exponents.

Property 1 *Given a real number x and integers a and b,*

$$x^a \cdot x^b = x^{a+b}.$$

Example 3

Write $4^7 \cdot 4^{30}$ with a single exponent.

Here we simply apply Property 1 to get:

$$4^7 \cdot 4^{30} = 4^{7+30} = 4^{37}.$$

Don't ask me what the number is though.

The next property we will get is a generalization of the following example.

Example 4

Write $(4^7)^3$ with a single exponent.

In order to simplify this expression, we simply write out what it means to raise something to an exponent and then see what we've got.

$$(4^7)^3 = (4^7) \cdot (4^7) \cdot (4^7) = 4^{7 \cdot 3} = 4^{21}.$$

Of course we can easily see how to make this more general.

Property 2 *For any real number x and integers a and b,*

$$(x^a)^b = x^{a\cdot b}.$$

Example 5

Write $(6^5)^{20}$ with a single exponent.

Here we simply apply Property 2 to get:

$$(6^5)^{20} = 6^{5\cdot 20} = 6^{100}.$$

Certainly $(3\cdot 2)^a = 6^a$ right? Let's play with this in an example to lead into the next property.

Example 6

Simplify $16\cdot 3^4$ to a power of 6.

$$\begin{aligned} 16\cdot 3^4 &= 2^4\cdot 3^4 \\ &= (2\cdot 2\cdot 2\cdot 2)\cdot(3\cdot 3\cdot 3\cdot 3) \\ &= (2\cdot 3)\cdot(2\cdot 3)\cdot(2\cdot 3)\cdot(2\cdot 3) \\ &= (2\cdot 3)^4 \\ &= 6^4. \end{aligned}$$

Property 3 *For any real numbers x and y and integer a,*

$$x^a\cdot y^a = (xy)^a.$$

Example 7

Simplify $(2x)^3(3x)^2$.

$$\begin{aligned} (2x)^3(3x)^2 &= 2^3\cdot x^3\cdot 3^2\cdot x^2 \\ &= 2^3\cdot 3^2\cdot x^3\cdot x^2 \\ &= 8\cdot 9\cdot x^{3+2} \\ &= 72x^5. \end{aligned}$$

It we can multiply, certainly we can divide too. Let's consider what happens when we divide.

Example 8

Simplify $\frac{2^4}{2^2}$.
This is easy. All we to is write it out and cancel things off.

$$\begin{aligned} \frac{2^4}{2^2} &= \frac{2\cdot 2\cdot 2\cdot 2}{2\cdot 2} \\ &= 2\cdot 2 \\ &= 2^2 \\ &= 4. \end{aligned}$$

What do we think 2^{-2} would mean then? Let's try the another example.

Example 9

Simplify $2^4 2^{-2}$.
Using Property 2 we get $2^4 2^{-2} = 2^{4+(-2)} = 2^{4-2} = 2^2 = 4$.

Notice this is the same as the previous example! So that means multiplying by 2^{-2} is the same as dividing by 2^2. This leads to our next property.

Property 4 *For any non-zero real number z and positive integer a,*

$$x^{-a} = \frac{1}{x^a}.$$

Example 10

Simplify $\frac{(2x)^4}{2^6 x^3}$.
Here we use Property 3 to break up the top and then Property 4 to simplify the expression.

$$\begin{aligned} \frac{(2x)^4}{2^6 x^3} &= \frac{2^4 x^4}{2^6 x^3} \\ &= \frac{x}{2^2} \\ &= x2^{-2}. \end{aligned}$$

In light of Property 4 and the previous example, we get the following.

Property 5 *For any real number $x \neq 0$ and positive integers a and b,*

$$\frac{x^a}{x^b} = x^{a=b}.$$

Example 11

Simplify $\left(\frac{x}{2}\right)^4$

$$\begin{aligned} \left(\frac{x}{2}\right)^4 &= \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x}{2}\right) \\ &= \frac{x\cdot x\cdot x\cdot x}{2\cdot 2\cdot 2\cdot 2} \\ &= \frac{x^4}{2^4} \\ &= \frac{x^4}{16} \end{aligned}$$

This example leads easily into the next property.

Property 6 *For any two real numbers x and y with $y \neq 0$ and a positive integer a,*

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}.$$

Our final property can be considered a definition but it does have motivation as we will see in subsequent examples.

Property 7 *For any real number x,*

$$x^1 = x, \qquad x^0 = 1.$$

Example 12

Simplify $\frac{4^{17}}{4^{17}}$.

First of all, for any nonzero number x, we know that $\frac{x}{x} = 1$ so this expression is equal to 1. The more interesting part is how this relates to Property 7. Using Property 5, we get:

$$\begin{aligned} 1 = \tfrac{4^{17}}{4^{17}} &= 4^{17-17} \\ &= 4^0 \end{aligned}$$

so we have shown that $4^0 = 1$ but there is nothing special about 4 in this example. We can do the same thing with any other number (except 0).

1.1.2 Scientific Notation

Exponents are used in science (and on your calculator if the answer doesn't fit on the screen) all the time. Almost anyone who has used a calculator for big numbers has gotten an answer like $1.3435624E14$. Here, the calculator is using scientific notation, a handy way to write large numbers in a small amount of space. Basically the idea is to make a notation that eliminates leading or trailing zeroes from a number. You'll understand more in a minute.

Definition 1 *A number x in* Scientific Notation *is written as:*

$$y \times 10^a$$

where $1 \leq y < 10$ and a is an integer.

The definition doesn't really tell us how to use the notation though. For that, we need a process for getting into and out of the notation.

Example 13

Write $10,200,000,000,000$ in scientific notation.

The idea is moving the decimal. Right now, the decimal is not written but the implied location is all the way at the right.

1. See how many places we need to move the decimal to put it just to the right of the first nonzero digit. In this case, we have to move it 13 places.

2. If we are moving the decimal a places to the left, we say $\times 10^a$. If we are moving it a places to the right, we say $\times 10^{-a}$. In this case, we're moving it 13 places to the left so we will say $\times 10^{13}$.

3. The value of x is what is left after we move the decimal. In this case, we simply get 1.02 since all the extra zeroes drop off as unnecessary.

Our result is then going to be 1.02×10^{13}.

Example 14

Write 0.00035064 in scientific notation.

We want to move the decimal 4 places to the right to get it just to the right of the first nonzero term. Hence we get 3.5064×10^{-4}.

Example 15

Write 2.320426×10^5 in standard notation.

Here we're trying to work backwards. We're multiplying by 10^5 so the decimal should move to the right 5 places. Therefore our answer should be $232,046.2$.

1.1.3 Monomials

Polynomials are the most basic of functions that are commonly studied. We begin, though, with the building blocks of these functions. Our pieces are called *monomials* or sometimes *terms*.

Example 16

The following are examples of monomials:

$$1, \quad 4x^2, \quad x, \quad t^7, \quad 9252x^3y^6t^9.$$

For a moment, let's consider the monomial $6x^4t^3$. Here the 6 is called a *coefficient* while x and t are both variables. In the case of the monomial x^4, the coefficient is assumed to be 1.

1.1.4 Polynomials

A polynomial is just a sum (or difference) of monomials.

Example 17

The following are examples of polynomials:

$$1, \quad 4x^2, \quad 2x^2 + x - 1, \quad 1.6x + 24.3x^7 - 435x^4y^3 - y.$$

Definition 2 *The* degree *of a polynomial (in one variable) is the largest exponent of the variable in any one monomial.*

Example 18

Find the degree of the following: (a) $5x^3 + 2x - 1$, (b) $t + 3t^6 - 24$, (c) 5.

(a) The largest exponent is in the first monomial $5x^3$ which is 3.

(b) This time we get the second monomial which gives us 6.

(c) Here there is no variable so the assumed degree is 0 (recall, this comes from the fact that $x^0 = 1$ for any variable x).

Addition and subtraction of polynomials is easy once you learn to use parentheses "( )" or sometimes "[ ]" effectively. They always mean that you're supposed to do what is inside first.

Example 19

Add the polynomials $4x^2 - 2x + 1$ and $3x^2 + 4x - 3$.

All we have to do here is group the terms that have similar degrees as follows:

$$\begin{aligned}(4x^2 - 2x + 1) + (3x^2 + 4x - 3) &= (4x^2 + 3x^2) + (-2x + 4x) + (1 - 3) \\ &= 7x^2 + 2x - 2.\end{aligned}$$

Property 8 *The* distributive *property states that if we multiply a whole collection of terms by a number, we get the same thing as if we multiplied each term by that number separately and then put them back together.*

This property really needs an example to be fully understood.

Example 20

Simplify $2(x^2 - 3x + 1)$.

According to the disributive property, this should be $2 \cdot x^2 - 2 \cdot 3x + 2 \cdot 1 = 2x^2 - 6x + 2$.

Now let's do a harder example.

Example 21

Simplify $2[x^2 - x^3 + 3(x-4) - (3x^2 + x^4 - 5)]$.

We always do what is in the innermost parentheses first. Here the first inner set we come to is $3(x-4)$. Using the process from the previous example, we get that this part is $3x - 12$. Now what about the next set. We have $... - (3x^2 + x^4 - 5)$. This is the same as $... + (-1)(3x^2 + x^4 - 5)$, right? Then all we have to do is distribute the "-1" to get $-3x^2 - x^4 + 5$ from that part.

Let's see what we've got so far. The second line is what we've done so far... after that, all we have to do is group the like terms and simplify.

$$\begin{gathered} 2[x^2 - x^3 + 3(x-4) - (3x^2 + x^4 - 5)] \\ = 2[x^2 - x^3 + 3x - 12 - 3x^2 - x^4 + 5] \\ = 2[-x^4 - x^3 - 2x^2 + 3x - 7] \\ = -2x^4 - 2x^3 - 4x^2 + 6x - 14. \end{gathered}$$

1.1.5 Exercises

1. Find:
   (a) 2^5,
   (b) 3^3,
   (c) 1^5,
   (d) $(0.5)^2$.

2. Write the following expression with a single exponent:
   (a) $3^5 \cdot 3^3$,
   (b) $x^4 \cdot x^{30}$,
   (c) $a^3 \cdot a^4 \cdot a^5$.

3. Write the following expression with a single exponent:
   (a) $(3^5)^3$,
   (b) $(x^4)^{30}$,
   (c) $((b^3)^2)^5$.

4. Simplify:
   (a) $(3x)^3(2x)^4$,
   (b) $(2x^2)^2(4x)$,
   (c) $(xy)^2x^3y^2$,
   (d) $(3xy^2)^3(2xy^3)^2$.

5. Write the following in scientific notation:
   (a) $1,213,209,630,000$,
   (b) 603.124,
   (c) 0.009346,
   (d) 0.191.

6. Write the following in standard notation:
   (a) 1.44632×10^7,

(b) 1.44632×10^4,
(c) 1.44632×10^{-4},
(d) 9.0001×10^{10}.

7. Find the degree of the following polynomials:
   (a) $3x^2 + 2x - 1$,
   (b) $5x^4 - x^7 + 1$,
   (c) $2t^9 - t$,
   (d) 9,
   (e) $4r - 5r^6$.

8. Add or subtract:
   (a) $(2x^3 - 2) + (4x^3 - x^2 + 5x)$,
   (b) $(3t + 4) - (2t^2 + t - 1)$.

9. Simplify:
   (a) $3[2x + 1 - (4x^2 + 3x) + 2(2x + 3)]$,
   (b) $2t^3 + t(2t^2 + 3)$.

1.2 Multiplying Polynomials

Today's lesson is all about the distributive property. Every time we see something multiplied by a polynomial in parentheses, we have to remember to distribute!

Example 1

Recall the distributive property. Find $3(2x+3)$.

$$\begin{aligned} 3(2x+3) &= 3\cdot 2x + 3\cdot 3 \\ &= 6x+9. \end{aligned}$$

There's no reason that we should only multiply by a number. Monomials work just as well.

Example 2

Find $2x(4x^3-3x+1)$.

This is another simple application of the distributive property. We just have to make sure the $2x$ hits each of the terms in the parentheses.

$$\begin{aligned} 2x(4x^3-3x+1) &= 2x\cdot 4x^3 - 2x\cdot 3x + 2x\cdot 1 \\ &= 8x^4-6x^2+2x. \end{aligned}$$

In fact, a polynomial works just as well too.

Example 3

Find $(t+2)(3t-4)$.

Again we apply the distributive property... but this time we do it twice. Break the question up into two blocks (delineated by the parentheses). We first distribute the first block into the second one (breaking the second block into monomials) making sure that the first block hits each part of the second. Finally, we distribute again, breaking the first block into pieces.

$$\begin{aligned} (t+2)(3t-4) &= (t+2)3t-(t+2)4 \\ &= t\cdot 3t + 2\cdot 3t - t\cdot 4 - 2\cdot 4 \\ &= 3t^2+6t-4t-8 \\ &= 3t^2+2t-8. \end{aligned}$$

Notice how we had to distribute the minus sign in there. The right side of the first line could also be written as $(t+2)3t+(t+2)(-4)$ which may make it easier to see where the negative sign should go.

1.2.1 FOIL

The *FOIL* method is only meant to be a (slightly) quicker way to multiply two binomials (polynomials with two terms). Suppose we have two binomials $(a+b)$ and $(c+d)$ and we want to multiply them together. We can do the above process to get:

$$\begin{aligned}(a+b)(c+d) &= (a+b)c + (a+b)d \\ &= ac + bc + ad + bd\end{aligned}$$

but we would like a way to do this a little quicker.

FOIL stands for First, Outer, Inner, Last. It tells us the pairings that we have to multiply together in the process. Using the above example, we examine what these mean.

First: ac takes the first term of each of the binomials and multiplies them together.

Outer: ad is the product of the outside terms when you write the product on one line.

Inner: bc is the product of the inner terms as above.

Last: bd takes the second (or last) term of each of the binomials and multiplies them together.

Example 4

Simplify $(x-1)(x+3)$ using the FOIL method.

First: x^2. Outer: $3x$. Inner: $-x$. Last: -3. Now all we have to do is add these up. We get:

$$x^2 + 3x - x - 3 = x^2 + 2x - 3.$$

Example 5

Let's repeat the example from above using FOIL. Find $(t+2)(3t-4)$.

First: $3t^2$. Outer: $-4t$. Inner: $6t$. Last: -8. So we add them up to get:

$$3t^2 - 4t + 6t - 8 = 3t^2 + 2t - 8.$$

Thankfully this is the same as we got before. Which was quicker?

1.2.2 General Polynomial Multiplication

Finally we discuss a method for multiplying pairs of larger polynomials together. The idea is just to make sure we cover all possible pairs of monomials.

Suppose we would like to multiply $(a+b+c)$ by $(d+e+f)$. In order to be sure we multiply every term of the first by every term of the second, create a table as follows. List all the terms of the first polynomial accross the top and all the terms of the second along the left side. In the table, put the product of the corresponding terms (see the table below). In the end, all we have to do is add up all the terms (joining like terms).

	a	b	c
d	ad	bd	cd
e	ae	be	ce
f	af	bf	cf

If this makes no sense at this point, you're doing just fine. Let's do some examples to get a better feel for the process.

Example 6

Multiply $(x^2 + 2x - 1)(2x^2 - 3x + 1)$.

Again we use the chart as before:

	x^2	$2x$	-1
$2x^2$	$2x^4$	$4x^3$	$-2x^2$
$-3x$	$-3x^3$	$-6x^2$	$3x$
1	x^2	$2x$	-1

We then add up the insides of the table to get:

$$2x^4 + 4x^3 - 2x^2 - 3x^3 - 6x^2 + 3x + x^2 + 2x - 1$$

and after we join like terms, the answer is

$$2x^4 + x^3 - 7x^2 + 5x - 1.$$

Discussion: Be careful with the negatives! Make sure to distribute the negative signs when we multiply. It may be a good idea to change every minus sign in the problem from $a - b$ to $a + (-b)$ to help follow the negative signs. Try some examples.

In Class: Have groups work on Ex. 4.

1.2.3 Exercises

1. Multiply:
   (a) $4(x^3 - 3)$.
   (b) $10(t + 10)$.
   (c) $2t(t^3 + 3t - 2)$.
   (d) $3y^2(x^2 + 5x - 3)$.

2. Use FOIL to multiply:
   (a) $(x + 1)(x - 3)$.
   (b) $(t^4 + 3t)(t^2 - 1)$.
   (c)* $(3xy^2 - 1)(2x^2x^3 - x^3y)$.

3. Multiply (being careful with the negative signs):
   (a) $-x(-x-2)$.
   (b) $-(x-1)(-x^2-3x)$.
   (c) $-x(x-2)(-x-x^2)$.
   (d) $-2(-x^2-x-1)(-y-xy)$.

4. * Multiply:
   (a) $2(2t^2-3t+6)(5t^2-t+1)$.
   (b) $3x(4x^4-2x^2+1)(3x^4+2x^2-1)$.
   (c) $2y(2x^3-3x+1)(3y^3+2y^2-y)$.

1.3 Factoring By Grouping

1.3.1 Greatest Common Factor

You may recall what the greatest common factor of two numbers is. It's the biggest number that divides into both of the numbers. Let's use the following example to demonstrate the concept.

Example 1 *Find the greatest common factor of* 12 *and* 20.

Certainly 2 divides both numbers but we also get 4. We can easily check that 4 is the biggest such number so 4 is the greatest common factor.

The following is a very natural extension of the idea of the greatest common factor.

Definition 3 *The* greatest common factor *of a polynomial is the largest monomial that divides each term of the polynomial.*

Let's do some easy examples to see how this works.

Example 2 *Find the greatest common factor of* $4x^3 + 2x^2$.

There are two parts to this process. First we see how many copies of x we can factor out. The smallest degree of the terms is 2 so we can pull x^2 from each term. Now the question is finding the biggest coefficient we can. This boils down to exactly the same problem we had before. Here we just find the greatest common factor of the coefficients. Between 4 and 6, the greatest common factor is, easily, 2. Hence, the greatest common factor of the polynomial $6x^3 + 4x^2$ is $2x^2$.

Example 3 *Find the greatest common factor of* $15a^4 - 6a^2 + 21a$.

Again we find the smallest power of the variable a which is 1. That means we can factor out an a from the whole thing. Then we begin looking at coefficients. What is the largest common factor of $15, 6$ and 21? That's going to be 3. Therefore, the greatest common factor of $15a^4 - 6a^2 + 21a$ is $3a$.

Definition 4 *To* factor *a polynomial is to break the polynomial into a product of* factors *(smaller polynomials).*

Essentially, factoring is going backwards through the processes we talked about in the previous section. In the last section, we discussed multiplying polynomials. Here, we are breaking a polynomial into pieces.

Definition 5 *The* factored form *of a polynomial is an expression of the polynomial in terms of a product of factors.*

Example 4 *Write the polynomial* $15a^4 - 6a^2 + 21a$ *from above in factored form.*

We already know that the greatest common factor of this polynomial is $3a$. The question now remains, what is left if we divide out $3a$? We have $5a^3$ from the first term, $-2a$ from the second and 7 from the last term. The factored form is therefore going to be $3a(5a^3 - 2a + 7)$.

Example 5 *Write the polynomial* $6x^2y^4 - 12x^3y^2$ *in factored form.*

First of all, we can take out a total of two xs and two ys. The greatest common factor of the coefficients is clearly 6 so the greatest common factor is $6x^2y^2$. Using the same removal process as above, we see that the factored form is $6x^2y^2(y^2 - 2x)$.

1.3.2 Factoring by Grouping

Now that we know what it means to factor and we can do some basic factoring, let's try some more difficult examples. In this section, we will factor by grouping. This means we will pair up monomials of a polynomial hoping to find some useful information. Let's just dive into some examples.

Example 6 *Factor* $x^3y^3 + 6x^3 + 2y^3 + 12$ *by grouping.*

The first observation we should make at this point is that the greatest common factor of this polynomial is 1 (meaning that we cannot factor out any monomial). We can, however, factor out an x^3 from the first two terms. Also, we can factor a 2 from the second two terms. This leaves us with

$$x^3(y^3 + 6) + 2(y^3 + 6).$$

Now we have two pieces, each of which has a common factor of $y^3 + 6$. We apply our knowledge of greatest common factors to reduce this to $(x^3+2)(y^3+6)$ which is the answer.

Example 7 *Factor* $2t^3 - 2t - 3t^2 + 3$ *by grouping.*

Here we factor $2t$ from the first two terms and -3 from the last two terms. If we are careful with the plusses and minuses, we get $2t(t^2-1)-3(t^2-1)$ which reduces down to $(2t - 3)(t^2 - 1)$.

The grouping will not always be so obvious, as seen in the following examples.

Example 8 *Factor* $2a^4 - 6b^4 + 4a^3b^3 - 3ab$ *by grouping.*

First we try to group the first two terms together. The greatest common factor here is 2... but we want to do better than that. Let's rearrange the terms so we can pull more things out at one time. Let's group the first and third terms and group the second and last terms together to get $2a^4 + 4a^3b^3 - 6b^4 - 3ab$.

Now we factor out the greatest common factors to get $2a^3(a+2b^3)-3b(2b^3+a)$. Hence the answer is $(2a^3 - 3b)(2b^3 + a)$.

Example 9 *Factor $x^2 + 3x + 2$ by grouping.*

There's nothing here to group! We need at least 4 terms to be able to group anything. We will, in the next section, see an easier way to factor this particular polynomial but let's try it by grouping. The technique is simply to break the middle term into two terms in such a way that things will group nicely and we can factor easily.

If we break $3x$ into $x + 2x$ then we get

$$x^2 + x + 2x + 2.$$

Now we can pull out x from the first two terms and 2 from the second pair to get $x(x + 1) + 2(x + 1)$ which clearly breaks down into $(x + 2)(x + 1)$.

1.3.3 Exercises

1. Find the greatest common factor of:
   (a) $7x^2 - 21x$,
   (b) $3xy^4 - 12x^2y^2 + 6x^3y$,
   (c)* $3x^4y^2 - 2y^3 + 7x^5$.

2. Factor the following by grouping:
   (a) $x^2y + x + 3xy + 3$,
   (b) $t^2 - rt - st + rs$,
   (c) $t^3 + 2t^2 - 8t - 16$,
   (d) $3x^3 + 6x^2 - 4x - 8$,
   (e)* $x^2 + 5x + 6$.

1.4 Factoring

1.4.1 Factoring Trinomials with Unit Leading Coefficient

We learned before how to multiply two binomials. Recall the process involved.

Example 1

Multiply the binomials $x+1$ and $x-4$.

For this question, we simply use the FOIL method as follows:

$$\begin{aligned}(x+1)(x-4) &= F+O+I+L \\ &= x^2-4x+x-4 \\ &= x^2-3x-4.\end{aligned}$$

More generally, we know how to multiply any pair of binomials of the form $ax+b$ and $cx+d$ where a, b, c and d are all numbers. For this section, we would like to focus on what happens when we multiply binomials of the form $x+a$ and $x+b$. What do we notice when we multiply these? Well, let's do it! We don't need to know what a and b are to actually multiply these binomials.

Example 2

Multiply the binomials $x+a$ and $x+b$.

Again we use the FOIL method to get:

$$\begin{aligned}(x+a)(x+b) &= x^2+bx+ax+ab \\ &= x^2+(a+b)x+ab.\end{aligned}$$

What do we notice about this result? The first term is always x^2. In general this is not always the case (as we've seen before) but if we assume the above conditions on the binomials, we will always get this result.

Hence, if we want to factor a trinomial with unit leading coefficient (meaning the x^2-term does not have a number in front of it), we just need to find the numbers a and b so that the above example works. Really that means we need to write the trinomial in the form $x^2+(a+b)x+ab$. Let's try some examples!

Example 3

Factor x^2+4x+4.

The goal is to make this trinomial fit our result from the previous example above. What two numbers a and b add up to 4 but also multiply to give you 4? This one is easy. The combination $a=2$ and $b=2$ works very well. How do we know this works though? Well, that's the easy part. We just go back and multiply the binomials to check our work.

$$\begin{aligned}(x+2)(x+2) &= x^2+2x+2x+4 \\ &= x^2+4x+4.\end{aligned}$$

This multiplication got us back to the original question so that means that $(x+2)(x+2) = (x+2)^2$ is our factored form.

How about another?

Example 4

Factor $r^2 + 2r - 3$.

Again we want to make this fit the form $r^2 + (a+b)r + ab$ but this time there is a minus sign in there. That just means that at least one of a or b must be negative... that doesn't hurt us. In fact, since the product is negative, that means that exactly one of a or b must be negative. So, what two numbers add up to 2 and multiply to -3? How about -1 and 3? Let's check.

$$\begin{aligned}(r-1)(r+3) &= r^2 + 3r - r - 3 \\ &= r^2 + 2r - 3\end{aligned}$$

which is exactly what we wanted.

Sometimes even if there is more involved, the same process still works.

Example 5

Factor $x^2 + 3xy + 2y^2$.

We need to find two monomials that, when added, give us $3y$ and when multiplied, give us $2y^2$. That's easy, what about y and $2y$. Let's try just to make sure:

$$\begin{aligned}(x+y)(x+2y) &= x^2 + 2xy + xy + 2y^2 \\ &= x^2 + 3xy + 2y^2.\end{aligned}$$

1.4.2 Factoring Other Trinomials

First we deal with a simple case in which there is a common factor of each term that we can pull out right from the start.

Example 6

Factor $3r^2 + 6r - 9$.

First notice that the leading coefficient is not 1 so we cannot apply the arguments from the previous subsection. We can, however, factor out a 3 from each term. When we do this, we get $3(r^2 + 2r - 3)$ and this is precisely the example we did above. Hence, the answer is $3(r-1)(r+3)$.

Let's try another one just to be sure.

Example 7

Factor $4t^2r + 20tr^2 + 16r^3$.

This time we first remove $4r$ from each term to be left with $t^2 + 5tr + 4r^2$. Two terms that sum up to $5r$ and multiply to $4r^2$ would be $4r$ and r so we get $4r(t + 4r)(t + r)$. You should also check this answer as above.

Hence, it is always best to factor out any common factors of the terms before proceeding.

What if there is not a common factor? Then the problem is more complicated.

Trial and Error

Suppose we want to factor a polynomial like $2x^2 - 5x - 3$. None of the previous techniques apply. One thing we do know though, is that the factors will be a pair of binomials where the first terms multiply to give $2x^2$ and the last terms multiply to give -3. There cannot be too many combinations that do that right? Let's list all of them. For the first terms we get $2x$ and x. The possibilities for the second terms are 3 and -1 or -3 and 1. Now we just try all combinations of these.

	Factors	First	Second	Third
1	$(2x + 3)(x - 1)$	$2x^2$	x	-3
2	$(2x - 1)(x + 3)$	$2x^2$	$5x$	-3
3	$(2x - 3)(x + 1)$	$2x^2$	$-x$	-3
4	$(2x + 1)(x - 3)$	$2x^2$	$-5x$	-3

The last try (number 4) works. Notice that all of the choices give us the correct first and last terms (that was how we constructed this list) but the only difference is in the middle terms. Notice also that we didn't need to consider negative versions of the first terms. It turns out, that if we did, we would only be repeating the same arguments and not getting any new information.

Example 8

Factor $6x^2 - 13x + 5$.

The possibilities for the first terms are x and $6x$ or $2x$ and $3x$. For the third terms, we have only 1 and 5 or -1 and -5. Hence, the possibilities for factors are:

$$(x + 1)(6x + 5)$$
$$(x + 5)(6x + 1)$$
$$(x - 1)(6x - 5)$$
$$(x - 5)(6x - 1)$$
$$(2x + 1)(3x + 5)$$
$$(2x + 5)(3x + 1)$$
$$(2x - 1)(3x - 5)$$
$$(2x - 5)(3x - 1)$$

of which only the 7^{th} gives us the correct second term.

Notice that this method gets very complicated and tedious as the number of factors of the first and last terms gets large.

Grouping

We saw a glimpse of how to use this method before. This method involves breaking the middle term into two parts and then appling the process of factoring by grouping.

This process can be broken down into the following steps. Given a trinomial $ax^2 + bx + c$, we:

- Let $d = ac$.
- Find a pair of numbers e and f with $ef = d$ and $e + f = b$.
- Rewrite the polynomial as $ax^2 + ex + fx + c$.
- Factor by grouping.

Example 9

Factor $2x^2 - 5x - 3$.

Notice that we have already solved this example in the previous subsection but let's see if this way is easier.

First let $d = 2 \times (-3) = -6$. Since $b = -5$, we need to find two numbers e and f such that $ef = -6$ and $e + f = -5$. Certainly we see that $e = 1$ and $f = -6$ works so we try to use grouping on $2x^2 + x - 6x - 3$. From the first pair of terms we can factor x and from the second pair of terms we factor -3. This results in $x(2x+1) - 3(2x+1)$ which gives us a factored form of $(x-3)(2x+1)$ which is exactly the answer that we got before.

1.4.3 Special Factoring

Perfect Squares

The point of this subsection is to study polynomials with the factored form $(ax + b)^2$ which result in polynomials looking like $a^2x^2 + 2abx + b^2$.

If we look at this carefully, we see that the following give us nice formulas for factoring some classes of polynomials:

$$\begin{aligned} a^2x^2 + 2abx + b^2 &= (ax + b)^2 \\ a^2x^2 - 2abx + b^2 &= (ax - b)^2 \end{aligned}$$

Example 10

Factor $r^2 - 6r + 9$.
By the second formula above, we get $(r-3)^2$.

Example 11

Factor $x^4 + 8x^2 + 16$.
By the first formula above (replacing x with x^2), we get $(x^2 + 4)^2$.

Example 12

Factor $4t^2 - 12t + 9$.
This time we use the second formula to get $(2t - 3)^2$.

Difference of Squares

In this subsection, we study the special structure of polynomials consisting of the difference between two squares. For example, we consider the structure of $a^2 - b^2$. This is special because it is the general form of the factors $(a+b)(a-b)$. Notice that the second term cancels off when we apply the FOIL method to multiply these factors, leaving us with only the squares.

Example 13

Factor $x^2 - 16$.
Here $a = x$ and $b = 4$ so we get the factored form of $(x + 4)(x - 4)$.

Example 14

Factor $4t^2 - 9r^4$.
In this case, $a = 2t$ and $b = 3r^2$ so we get the factored form of $(2t+3r^2)(2t-3r^2)$.

Sum and Difference of Cubes

For this subsection, we use the following two observations.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

and

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Verify these formulas by multiplying the terms on the right to get the result on the left. These are very handy in factoring polynomials in these forms.

Example 15

Factor $8x^3 - 21$.

This is the difference of two cubes where $a = 2x$ and $b = 3$ so we get the factored form of $(2x - 3)(4x^2 + 6x + 9)$.

Example 16

Factor $x^6 - y^6$.

We first notice that this is a difference of squares so we get $(x^3 - y^3)(x^3 + y^3)$. Then we apply the sum and difference of cubes results from above to get:

$$(x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2).$$

1.4.4 Exercises

1. Multiply the following:
   (a) $(x - 1)(x - 3)$,
   (b) $(x - 2y)(x + y)$.

2. Factor the following:
   (a) $x^2 - 4x + 3$,
   (b) $(x^2 - yx - 2y^2$,
   (c) $t^2 - t - 12$,
   (d) $s^2 + s - 12$.

3. Factor the following:

4. Factor the following:
   (a) $y^2 + 3y + \frac{9}{4}$,
   (b) $64 + 16t + t^2$.

5. Factor the following:
   (a) $81r^2 - 25$,
   (b) $36x^2y^4 - 16t^4y^2$.

6. Factor the following:

1.5 Using Factoring

Why have we been doing all this factoring? What is it used for? Let's do lots of examples. But first, a definition.

Definition 6 *Any equation that can be written in the form*

$$ax^2 + bx + c = 0$$

is called a quadratic equation *and the above form is called the* standard form *of a quadratic equation.*

Naturally, for given values of a, b and c, we would like to find what values of x will satisfy this equation. Using the fact that $r = 0$ or $s = 0$ if and only if $rs = 0$, we see that the above equation holds if and only if at least one of the factors of the polynomial is zero. That's where factoring comes in.

Example 1

Solve $x^2 - 5x - 6 = 0$.

The first goal is to factor the left side of this equation. This results in

$$(x - 6)(x + 1) = 0$$

so at least one of $(x + 1)$ or $(x - 6)$ must equal 0. Certainly they cannot both equal 0 at the same time so we get $x = 6$ and $x = -1$ as the solutions to this equation.

Example 2

Solve $\frac{1}{3}x^3 = \frac{5}{6}x^2 + \frac{1}{2}x$.

The first thing we have to do is to get this into standard form. We also don't like messing with fractions so let's multiply both sides of the equation by 6 to get rid of the fractions. This first step gives us

$$2x^3 = 5x^2 + 3x.$$

Next we move everything to one side of the equation. This gives us

$$2x^3 - 5x^2 - 3x = 0.$$

Now we factor out the greatest common factor (which is x) to get

$$x(2x^2 - 5x - 3) = 0$$

and we have something in standard form. Granted, there is still an x on the outside. We may just apply the above observation about products equalling zero to say that $x = 0$ is one of the solutions and proceed in finding the others.

We factor the parenthesized quadratic function to get

$$x(2x+1)(x-3) = 0$$

which means that $x = 0$, $x = 3$ and $x = -\frac{1}{2}$ are all roots of this equation.

Example 3

Find two consecutive positive integers for which the sum of these squares of the integers is 41.

We want to find two consecutive integers x and $x+1$ such that $x^2+(x+1)^2 = 41$. This looks a lot like a quadratic equation so let's try to put it in standard form.

$$\begin{aligned} x^2 + (x+1)^2 &= 41 \\ x^2 + (x^2 + 2x + 1) &= 41 \\ 2x^2 + 2x + 1 &= 41 \\ 2x^2 + 2x - 40 &= 0. \end{aligned}$$

Now all we have to do is factor this and see what we get. First we can divide both sides by 2 to get $x^2 + x - 20 = 0$ and the left side factors down to $(x+5)(x-4)$ so the solutions are $x = -5$ and $x = 4$. Since the question asked for positive integers, we get $x = 4$ and $x + 1 = 5$.

Before we're truly finished, we should go back and make sure we're fully answering the question and check our answer. The question asks for two consecutive positive integers (we provided 4 and 5) such that the sum of the squares ($4^2 + 5^2 = 16 + 25$) equals 41 (which is true).

Theorem 1 (Pythagorean Theorem) *For any right triangle with side lengths a, b and c (where a and b are incident to the right angle) we have:*

$$a^2 + b^2 = c^2.$$

Example 4

The lengths of the three sides of a right triangle are given by three consecutive integers. Find these integers.

First we want to draw a picture. Note that in a right triangle, the hypotenuse is always the longest side. Hence, we let x be the length of the hypotenuse and let $x - 1$ and $x - 2$ be the lengths of the other sides. We can now set up the equation that we need to solve.

$$(x-1)^2 + (x-2)^2 = x^2$$

The first step is, of course, to square the pieces on the left. This gives us:

$$(x^2 - 2x + 1) + (x^2 - 4x + 4) = x^2$$

and when we solve for 0, we get:

$$x^2 - 6x + 5 = 0.$$

Now all we have to do is find x. We can factor this quadratic into $(x-5)(x-1)$ which tells us that $x = 5$ or $x = 1$. If $x = 1$, then $x - 2 = -1$ and that doesn't make any sense. Our answer must be $x = 5$ so the sides of the triangle are $3, 4$ and 5. Let's check and make sure that works.

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2.$$

1.5.1 Exercises

1. There is a 13 foot pole leaning against a house. If the base of the pole is 5 feet away from the base of the house, how high up the house is the top of the pole?

Chapter 2

Rational Functions

2.1 Reducing and Division

Recall that multiplying or dividing the numerator and denominator of a fraction by the same thing always yields an equal expression. In math, this means that $\frac{a}{b} = \frac{ac}{bc}$.

A rational number (or fraction) is given by $\frac{a}{b}$ where a and b are integers. A *rational expression* is given by $\frac{A}{B}$ where A and B are polynomials. In the same way we reduce regular fractions (by dividing out the greatest common factor), we can reduce rational expressions.

Property 9 *If P, Q and K are polynomials with $Q, K \neq 0$ then*

$$\frac{P}{Q} = \frac{PK}{QK} \qquad \textit{and} \qquad \frac{P}{Q} = \frac{P/K}{Q/K}.$$

Example 1

Reduce: $\frac{9}{3}$.

We first find the greatest common factor of 3 and 9 (which is 3). Then we divide that out of the top and the bottom to get:

$$\frac{9}{3} = \frac{9/3}{3/3} = \frac{3}{1} = 3.$$

Example 2

Reduce: $\frac{x^2 - x}{2x^x + 4x}$.

Again, we find the greatest common factor of the top and the bottom. The top can be factored into $x(x-1)$ while the bottom can be factored into $2x(x+2)$. The only thing these have in common is x so the reduced form of this fraction is going to be:

$$\frac{x^2 - x}{2x^x + 4x} = \frac{(x^2 - x)/x}{(2x^x + 4x)/x} = \frac{x - 1}{2x + 4}.$$

Example 3

Reduce: $\frac{x^2-4}{x+2}$.

We cannot factor the bottom so we focus on the top. The top is a difference of squares so we factor this easily into $(x - 2)(x + 2)$. Then we get:

$$\frac{x^2 - 4}{x + 2} = \frac{(x - 2)(x + 2)}{x + 2} = \frac{x - 2}{1} = x - 2.$$

Example 4

Reduce: $\frac{x^2-9}{3-x}$.

Again there is nothing we can immediately do to the bottom so let's focus on the top. Again it is a difference of squares so we get $(x - 3)(x + 3)$. This looks disturbingly similar to the bottom... except not quite. Let's rearrange the bottom so that it looks even more similar.

$$\frac{x^2 - 9}{3 - x} = \frac{(x - 3)(x + 3)}{-x + 3}.$$

Now we play a little trick. Let's multiply the bottom by -1. Then we get:

$$\frac{(x - 3)(x + 3)}{-x + 3} = \frac{(x - 3)(x + 3)}{-1(x - 3)}.$$

Now something cancels off! Using this, we get:

$$\frac{(x - 3)(x + 3)}{-1(x - 3)} = \frac{x + 3}{-1} = \frac{-1(x + 3}{(-1)(-1)} = \frac{-x - 3}{1} = -x - 3.$$

2.1.1 Rational Functions

A rational *function* is exactly like a rational expression except used to give an output y in terms of the input x. A rational function looks like:

$$f(x) = \frac{P(x)}{Q(x)}.$$

where $P(x)$ and $Q(x)$ are polynomials.

Just like any functions, one of the first things we want to find is the domain of a rational function. Recall that the domain is defined to be the set of values for x that we can plug into the function in order to get out a meaningful answer y. By meaningful, I mean we can actually find a value of y.

First note, the domain of any polynomial is the entire set of real numbers. Basically, that means we can plug in any value for x and get a corresponding value for y.

So the question at hand is the following: if we divide two polynomials (to get a rational function), what can go wrong that would make us not have a domain of all real numbers? Really, when does division cause problems? What were we always told not to divide? Answer: You cannot divide anything by zero! So really we have to say that $Q(x) \neq 0$ to make the above meaningful.

Theorem 2 *The domain of a rational function is all real numbers* except *the values for which the denominator is zero.*

Example 5

Find the domain of the following:

$$f(x) = \frac{2x+1}{x-3}.$$

This is a rational function so the domain is all real numbers except where the denominator is zero. So the only thing we need to ask ourselves is where the denominator is zero. If $x - 3 = 0$, that means $x = 3$. Hence, the domain is "All reals except 3".

Example 6

Find the domain of $f(r)$:

$$f(r) = \frac{1}{r^2 + 4r + 4}.$$

For this problem, we again simply find where the bottom equals zero. Hence, we need to solve $r^2 + 4r + 4 = 0$. Here we just factor to get $(r + 2)(r + 2) = 0$ so only $r = -2$ satisfies the equation. Hence, the domain is all reals except -2.

Example 7

Find the domain of $g(x)$:

$$g(x) = \frac{x^2 - 16}{x+4}.$$

This time it's easy to see that the domain is all reals except -4 but there is one little difference. What if we factor the top? We get:

$$\frac{x^2-16}{x+4} = \frac{(x-4)(x+4)}{x+4} = x - 4.$$

The problem is that $x + 4$ is a polynomial and we know that the domain of this is all reals. How does that work? It turns out that the equality above is not quite true when we work with functions. Actually $g(x) = x - 4$ everywhere except when $x = -4$. At that one point, the function $g(x)$ is not defined.

2.1.2 Division

Alright, we've seen how to factor things and, in rational expressions, cancel off terms that are identical from the top and the bottom. The problem is, what if we want to just do basic long division on polynomials?

Let's recall an example of division.

Example 8

If $f(x) = 2x - 5$, divide $\frac{f(x)-f(a)}{x-a}$.

First we just plug things in, then cancel off everything we can... and then pray.

$$\begin{aligned} \frac{f(x)-f(a)}{x-a} &= \frac{(2x-5)-(2a-5)}{x-a} \\ &= \frac{2x-5-2a+5}{x-a} \\ &= \frac{2x-2a}{x-a} \\ &= \frac{2(x-a)}{x-a} \\ &= 2. \end{aligned}$$

Let's do one of the examples from above by using long division.

Example 9

Divide:

$$\frac{x^2 - 16}{x + 4}.$$

See the book for a better explanation.

All we have to do is look at the leading terms of each polynomial. How do we make the leading term of the bottom x turn into the leading term on top x^2. Certainly we have to multiply by x. That means we will subtract $x(x + 4) = x^2 + 4x$ from the top to get $-4x - 16$. Now how do we get from the leading term of the bottom (still x) to the new leading term on the top $-4x$. Multiply by -4 of course. This time we subtract $-4x - 16$ from the top which leaves us with zero. Hence, the result is $x - 4$.

2.1.3 Exercises

1.

2.2 Combining Rational Expressions

In this section, we discuss multiplying and adding rational expressions.

2.2.1 Multiplication

We multiply rational expressions in exactly the same way as rational numbers.

$$\frac{P(x)}{Q(x)} \cdot \frac{R(x)}{S(x)} = \frac{P(x)R(x)}{Q(x)S(x)}.$$

Let's do a couple examples.

Example 1

Multiply:

$$\frac{2x}{23xy^2} \cdot \frac{3y^3}{x^3y}.$$

As above, we just multiply straight accross. Finally, we simplify to make our answer look better.

$$\frac{2x^2}{23y^2} \cdot \frac{3y^3}{x^3} = \frac{6x^2y^3}{23x^3y^2} = \frac{6y}{23x}.$$

Example 2

Multiply:

$$\frac{x-2}{x+1} \cdot \frac{x^2+3x-1}{3x-2}.$$

Again we multiply straight accross but this time we have polynomials so we have to make sure to multiply every combination of terms.

$$\frac{x-2}{x+1} \cdot \frac{x^2+3x-1}{3x-2} = \frac{(x-2)(x^2+3x-1)}{(x+1)(3x-2)} = \frac{x^3+x^2-7x+2}{3x^2+2x-2}.$$

Example 3

Divide:

$$\frac{x-2}{x+1} \div \frac{3x-2}{x^2+3x-1}.$$

In order to divide two fractions, we simply recall that the division of a by b is the same as multiplying a by the reciprocal of b. Hence, if we flip the fraction on the right, we get exactly the example that we just did above and the answer is identical.

2.2.2 Addition

Just like in the case of rational numbers, in order to add two rational expressions, we first have to find a common denominator. When the denominator is a polynomial, that can sometimes be tricky. Let's do some examples.

Example 4

Add:

$$\frac{x^2-4x+3}{x-1}+\frac{2x^2+4x+5}{x-1}.$$

This time we're lucky and the denominators are the same already. Hence, we just add the tops (leaving the bottoms alone) to get:

$$\frac{x^2-4x+3}{x-1}+\frac{2x^2+4x+5}{x-1}=\frac{(x^2-4x+3)+(2x^2+4x+5)}{x-1}=\frac{3x^2+8}{x-1}.$$

Example 5

Add:

$$\frac{2x-2}{x^2+4x+3}+\frac{x-1}{x^2+5x+6}.$$

In order to find the least common denominator, we have to factor the bottoms of both fractions. For the first one we get:

$$x^2+4x+3=(x+3)(x+1)$$

and for the second we get:

$$x^2+5x+6=(x+2)(x+3).$$

Hence, they already have $x+3$ in common so we only need to multiply the first fraction by $\frac{x+2}{x+2}$ and the second by $\frac{x+1}{x+1}$ to get common denominators.

$$\begin{aligned}
\frac{2x-2}{x^2+4x+3}+\frac{x-1}{x^2+5x+6} &= \frac{2x-2}{(x+3)(x+1)}+\frac{x-1}{(x+3)(x+2)} \\
&= \frac{(2x-2)(x+2)}{(x+3)(x+1)(x+2)}+\frac{(x-1)(x+1)}{(x+3)(x+2)(x+1)} \\
&= \frac{(2x-2)(x+2)+(x-1)(x+1)}{(x+3)(x+1)(x+2)} \\
&= \frac{(2x^2+2x-4)+(x^2-1)}{(x+3)(x+1)(x+2)} \\
&= \frac{3x^2+2x-5}{(x+3)(x+1)(x+2)} \\
&= \frac{(3x+5)(x-1)}{(x+3)(x+1)(x+2)}
\end{aligned}$$

You can leave the answer in factored form if the question (like this one) does not specify the desired form of the answer.

There is no difference between addition and subtraction... except for the subtraction part. You still have to find a common denominator first.

Example 6

Subtract $t - \frac{2}{t}$

First we have to get a common denominator. The first term is not a fraction though. What do we do? Actually, anything is a fraction if we want it to be. $t = \frac{t}{1}$. Then we just multiply the first term by $\frac{t}{t}$ to get:

$$\begin{aligned} t - \tfrac{2}{t} &= \tfrac{t}{t}t - \tfrac{2}{t} \\ &= \tfrac{t^2}{t} - \tfrac{2}{t} \\ &= \tfrac{t^2-2}{t}. \end{aligned}$$

Example 7

Add the reciprocals of two consecutive integers (and reduce).

Let's consider two consecutive integers x and $x + 1$. We want to add the reciprocals which are $\frac{1}{x}$ and $\frac{1}{x+1}$.

$$\begin{aligned} \tfrac{1}{x} + \tfrac{1}{x-1} &= \tfrac{1}{x}\tfrac{(x+1)}{(x+1)} + \tfrac{1}{x+1}\tfrac{(x)}{(x)} \\ &= \tfrac{x+1}{x^2+x} + \tfrac{x}{x^2+x} \\ &= \tfrac{2x+1}{x^2+x}. \end{aligned}$$

2.2.3 Exercises

1.

2.3 More Rational Expressions

What if we have rational expressions divided by rational expressions?

2.3.1 Complex Fractions

We now discuss fraction of fractions. Let's start with an example that should be rather familiar by this point.

Example 1

Simplify:

$$\frac{\frac{1}{2}}{\frac{3}{4}}$$

As we have seen before, we simply multiply the top by the reciprocal of the bottom to get:

$$\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2}\frac{4}{3} = \frac{4}{6} = \frac{2}{3}.$$

Now for something quite a bit more complicated.

Example 2

Simplify:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}.$$

Remember, the main goal here is to get rid of the fractions on the top and bottom. In this case, if we want to use the method above, we would have to find common denominators on the top and bottom to be able to add the fractions and then multiply to simplify the whole thing.

Let's try something easier. How can we get rid of both fractions on the top? Well, we could multiply by xy to get rid of both of them. The problem is, we can never just multiply by xy... unless we do the same thing to the bottom of the fraction. Lo and behold, that gets rid of the fractions there too! Hence, we get:

$$\begin{aligned}
\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}} &= \frac{(\frac{1}{x}+\frac{1}{y})xy}{(\frac{1}{x}-\frac{1}{y})xy} \\
&= \frac{\frac{xy}{x}+\frac{xy}{y}}{\frac{xy}{x}-\frac{xy}{y}} \\
&= \frac{y+x}{y-x}.
\end{aligned}$$

As if that was not exciting enough, let's try another.

Example 3

Simplify:

$$\frac{1-\frac{1}{a}}{1+\frac{1}{a}}.$$

For this, we again multiply by something on the top and bottom to get rid of both problems.

$$\frac{1-\frac{1}{a}}{1+\frac{1}{a}} = \frac{(1-\frac{1}{a})a}{(1+\frac{1}{a})a} = \frac{a-1}{a+1}.$$

Example 4

Simplify:

$$\frac{\frac{2x^2}{5y^3}}{\frac{4xy^3}{7}}.$$

For this example, we revert back to the original method which means multiply by the reciprocal.

$$\frac{\frac{2x^2}{5y^3}}{\frac{4xy^3}{7}} = \frac{2x^2}{5y^3}\frac{7}{4xy^3} = \frac{7x}{10y^6}.$$

2.3.2 Equations with Rationals

In order to solve equations involving rationals, the first step is to find a common denominator for all pieces of the puzzle.

Example 5

Solve for x:

$$\frac{x}{2} - 1 = x + \frac{2}{3}.$$

The least common denominator of each term of this equation is 6 so we multiply everything by 6 to get:

$$3x - 6 = 6x + 4.$$

Then we just rearrange the terms to get $3x = -10$ or $x = \frac{-10}{3}$. The most important thing, at this point, is to go back and make sure this answer works. Hence, we plug $x = \frac{-10}{3}$ back into the original equation and try to make sure everything works out. In this case, we should be ok.

Example 6

Solve for t:

$$\frac{t}{t-3} + \frac{1}{2} = \frac{3}{t-3}.$$

Here the common denominator is $2(t-3)$ or $2t-6$. Multiplying by this gives us:

$$2t + (t-3) = 6.$$

Then, after rearranging a little bit, we get $3t = 9$ or $t = 3$. This time, when we try to plug the answer back into the original equation, we see that the first term on the left and the only term on the right are undefined (division by zero). That means there is **no solution** to this example.

Example 7

Solve for y:

$$\frac{-5}{y} + \frac{2}{y^2} = -2.$$

This time our least common denominator is y^2 so when we multiply we get:

$$-5y + 2 = -2y^2$$

This means that we have a quadratic equation in the form of:

$$2y^2 - 5y + 2 = 0.$$

Factoring gives us $(2y-1)(y-2)$ so our possible answers come out to be $y = \frac{1}{2}$ and $y = 2$. Since $y = 0$ is the only possible answer that would cause trouble in the original equation, we know these answers are valid but you should still check them just to be sure.

Example 8

Solve for x:

$$y = \frac{x-3}{x-2}.$$

This is a little different from the previous questions. Here, we will not end up with numbers but another function... in terms of y this time.

First we find the least common denominator. That would be $x-2$. Multiplying this on both sides gives us $y(x-2) = x-3$ or $xy - 2y = x - 3$. The goal is to solve for x so let's group the two x terms together. That gives us

$xy - x = 2y - 3$. Factoring out the x gives us $x(y-1) = 2y-3$. Finally we just divide both sides by $y-1$ to get our final result of:

$$x = \frac{2y-3}{y-1}.$$

2.3.3 Exercises

1.

2.4 Applications of Rationals

Chapter 3

Rational Exponents

Chapter 4

Quadratics

Appendix A

Notation

- $\sim$ means "is approximately". This is used to replace = when the two things are not really equal but at least close. (This is NOT standard notation in math.)
- $\mathscr{C}$ means this exercise may require the use of a calculator.
- * means this exercise may be more difficult than most others.
- $\Leftrightarrow$ or "iff" may be used as short for "if and only if" for showing necessary and sufficient conditions.