JORDAN CANONICAL FORM WITH APPLICATION TO SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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JCF and DEs

A. Jordan Canonical Form

In order to understand and be able to use Jordan Canonical Form, we must introduce a new concept, that of a generalized eigenvector.

Definition A.1. *If $v \neq 0$ is a vector such that, for some λ,*

$$(A - \lambda I)^k(v) = 0$$

for some positive integer k, then v is a generalized eigenvector of A associated to the eigenvalue λ. The smallest k with $(A - \lambda I)^k(v) = 0$ is the index of the generalized eigenvector v.

Let us note that if v is a generalized eigenvector of index 1, then

$$(A - \lambda I)(v) = 0$$
$$(A)v = (\lambda I)v$$
$$Av = \lambda v$$

and so v is an (ordinary) eigenvector.

Recall that, for an eigenvalue λ of A, E_λ is the eigenspace of λ,

$$E_\lambda = \{v \mid Av = \lambda v\} = \{v \mid (A - \lambda I)v = 0\}.$$

We let $\tilde{E}_\lambda$ denote the generalized eigenspace of λ,

$$\tilde{E}_\lambda = \{v \mid (A - \lambda I)^k(v) = 0 \quad \text{for some } k\}.$$

It is easy to check that $\tilde{E}_\lambda$ is a subspace.

Since every eigenvector is a generalized eigenvector, we see that

$$E_\lambda \subseteq \tilde{E}_\lambda.$$

The following result (which we shall not prove) is an important fact about generalized eigenspaces.

Proposition A.2. *Let λ be an eigenvalue of A of multiplicity m. Then $\tilde{E}_\lambda$ is a subspace of dimension m.*

Example A.3. *Let A be the matrix* $A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$. *Then, as you can check, if* $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, *then* $(A - 2I)u = 0$, *so u is an eigenvector of A with associated eigenvalue 2 (and hence a generalized eigenvector of index 1 of A with associated eigenvalue 2). On the other hand, if* $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, *then* $(A - 2I)^2 v = 0$ *but* $(A - 2I)v \neq 0$, *so v is a generalized eigenvector of index 2 of A with associated eigenvalue 2.*

In this case, as you can check, the vector u is a basis for the eigenspace E_2, so $E_2 = \{ cu \mid c \in \mathbb{R} \}$ *is 1-dimensional.*

On the other hand, u and v are both generalized eigenvectors associated to the eigenvalue 2, and are linearly independent (the equation $c_1 u + c_2 v = 0$ only has the solution $c_1 = c_2 = 0$, as you can readily check), so $\tilde{E}_2$ has dimension at least 2. Since $\tilde{E}_2$ is a subspace of $\mathbb{R}^2$, it must have dimension exactly 2, and $\tilde{E}_2 = \mathbb{R}^2$ (and $\{ u, v \}$ is indeed a basis for $\mathbb{R}^2$).

Let us next consider a generalized eigenvector v_k of index k associated to an eigenvalue λ, and set

$$v_{k-1} = (A - \lambda I)v_k.$$

We claim that v_{k-1} is a generalized eigenvector of index $k-1$ associated to the eigenvalue λ. To see this, note that

$$(A - \lambda I)^{k-1} v_{k-1} = (A - \lambda I)^{k-1}(A - \lambda I)v_k = (A - \lambda I)^k v_k = 0$$

but

$$(A - \lambda I)^{k-2} v_{k-1} = (A - \lambda I)^{k-2}(A - \lambda I)v_k = (A - \lambda I)^{k-1} v_k \neq 0.$$

Proceeding in this way, we may set

$$\begin{aligned} v_{k-2} &= (A - \lambda I)v_{k-1} = (A - \lambda I)^2 v_k \\ v_{k-3} &= (A - \lambda I)v_{k-2} = (A - \lambda I)^2 v_{k-1} = (A - \lambda I)^3 v_k \\ &\vdots \\ v_1 &= (A - \lambda I)v_2 = \cdots = (A - \lambda I)^{k-1} v_k \end{aligned}$$

and note that each v_i is a generalized eigenvector of index i associated to the eigenvalue λ. A collection of generalized eigenvectors obtained in this way gets a special name.

Definition A.4. *If* $\{ v_1, \ldots, v_k \}$ *is a set of generalized eigenvectors associated to the eigenvalue* λ *of* A*, such that* v_k *is a generalized eigenvector of index* k *and also*

$$\begin{aligned} v_{k-1} =& (A - \lambda I)v_k, \quad v_{k-2} = (A - \lambda I)v_{k-1}, \quad v_{k-3} = (A - \lambda I)v_{k-2}, \\ & \cdots, \quad v_2 = (A - \lambda I)v_3, \quad v_1 = (A - \lambda I)v_2, \end{aligned}$$

then $\{ v_1, \ldots, v_k \}$ *is called a* *chain* *of generalized eigenvectors of length* k*. The vector* v_k *is called the* *top* *of the chain and the vector* v_1 *(which is an ordinary eigenvector) is called the* *bottom* *of the chain.*

Example A.5. *Let us return to example 3. We saw there that* $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *is a generalized eigenvector of index 2 of* $A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ *associated to the eigenvalue 2. Let us set*

$$v_2 = v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then

$$v_1 = (A - 2I)v_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

is a generalized eigenvector of index 1 (i.e., an ordinary eigenvector), and $\{ v_1, v_2 \}$ *is a chain of length 2.*

Remark A.6. It is important to note that a chain of generalized eigenvectors $\{ v_1, \ldots, v_k \}$ is entirely determined by the vector v_k at the top of the chain. For once we have chosen v_k, there are no other choices to be made: the vector v_{k-1} is determined by the equation $v_{k-1} = (A - \lambda I)v_k$; then the vector v_{k-2} is determined by the equation $v_{k-2} = (A - \lambda I)v_{k-1}$; etc.

With this concept in hand, let us return to Jordan Canonical Form. As we have seen, a matrix J in Jordan Canonical Form has a number of blocks $B_1, B_2, \ldots, B_m$, called Jordan blocks, along the diagonal. Let us begin our analysis with the case when J consists of a single Jordan block. So suppose J is a k-by-k matrix

$$J = \begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & 0 & \\ & & \lambda & 1 & & \\ & & & \ddots & \ddots & \\ & 0 & & & \lambda & 1 \\ & & & & & \lambda \end{bmatrix}.$$

Then

$$J - \lambda I = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}.$$

Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$

Then direct calculation shows:

$$\begin{aligned} (J - \lambda I)e_k &= e_{k-1} \\ (J - \lambda I)e_{k-1} &= e_{k-2} \\ &\vdots \\ (J - \lambda I)e_2 &= e_1 \\ (J - \lambda I)e_1 &= 0 \end{aligned}$$

and so we see that $\{ e_1, \ldots, e_k \}$ is a chain of generalized eigenvectors. We also note that $\{ e_1, \ldots, e_k \}$ is a basis for $\mathbb{R}^k$, and so

$$\tilde{E}_\lambda = \mathbb{R}^k.$$

We first see that the situation is very analogous when we consider any k-by-k matrix with a single chain of generalized eigenvectors of length k.

Proposition A.7. *Let $\{ v_1, \ldots, v_k \}$ be a chain of generalized eigenvectors of length k associated to the eigenvalue λ of a matrix A. Then $\{ v_1, \ldots, v_k \}$ is linearly independent.*

Proof. Suppose we have a linear combination

$$c_1v_1 + c_2v_2 + \cdots + c_{k-1}v_{k-1} + c_kv_k = 0.$$

We must show each $c_i = 0$.

By the definition of a chain, $v_{k-i} = (A-\lambda I)^i v_k$ for each i, so we may write this equation as

$$c_1(A-\lambda I)^{k-1}v_k + c_2(A-\lambda I)^{k-2}v_k + \cdots + c_{k-1}(A-\lambda I)v_k + c_kv_k = 0.$$

Now let us multiply this equation on the left by $(A-\lambda I)^{k-1}$. Then we obtain the equation

$$c_1(A-\lambda I)^{2k-2}v_k + c_2(A-\lambda I)^{2k-3}v_k + \cdots + c_{k-1}(A-\lambda I)^k v_k + c_k(A-\lambda I)^{k-1}v_k = 0.$$

Now $(A-\lambda I)^{k-1}v_k = v_1 \neq 0$. However, $(A-\lambda I)^k v_k = 0$, and then also $(A-\lambda I)^{k+1}v_k = (A-\lambda I)(A-\lambda I)^k v_k = (A-\lambda I)(0) = 0$, and then similarly $(A-\lambda I)^{k+2}v_k = 0, \ldots,$ $(A-\lambda I)^{2k-2}v_k = 0$, so every term except the last one is zero and this equation becomes

$$c_kv_1 = 0.$$

Since $v_1 \neq 0$, this shows $c_k = 0$, so our linear combination is

$$c_1v_1 + c_2v_2 + \cdots + c_{k-1}v_{k-1} = 0.$$

Repeat the same argument, this time multiplying by $(A-\lambda I)^{k-2}$ instead of $(A-\lambda I)^{k-1}$. Then we obtain the equation

$$c_{k-1}v_1 = 0.$$

and, since $v_1 \neq 0$, this shows that $c_{k-1} = 0$ as well. Keep going to get

$$c_1 = c_2 = \cdots = c_{k-1} = c_k = 0,$$

so $\{ v_1, \ldots, v_k \}$ is linearly independent. □

Theorem A.8. *Let A be a k-by-k matrix and suppose that $\mathbb{R}^k$ has a basis $\{ v_1, \ldots, v_k \}$ consisting of a single chain of generalized eigenvectors of length k associated to an eigenvalue a. Then*

$$A = PJP^{-1}$$

where

$$J = \begin{bmatrix} a & 1 & & & & \\ & a & 1 & & & \\ & & a & 1 & & \\ & & & \ddots & \ddots & \\ & & & & a & 1 \\ & & & & & a \end{bmatrix}$$

is a matrix consisting of a single Jordan block and

$$P = \left[v_1 \middle| v_2 \middle| \dots \middle| v_k \right]$$

is a matrix whose columns are generalized eigenvectors forming a chain.

Proof. Let P be the given matrix. We will first show by direct computation that $AP = PJ$.

It will be convenient to write

$$J = \left[j_1 \middle| j_2 \middle| \dots \middle| j_k \right]$$

and we see that j_i, the $i-th$ column of J, is the vector

$$j_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ a \\ 0 \\ \vdots \end{bmatrix}$$

with 1 in the $(i-1)-st$ position, a in the $i-th$ position, and 0 elsewhere.

We show that $AP = PJ$ by showing that their corresponding columns are equal.

Now

$$AP = A\left[v_1 \middle| v_2 \middle| \dots \middle| v_k \right]$$

so the $i-th$ column of AP is Av_i. But

$$\begin{aligned} Av_i &= (A - aI + aI)v_i \\ &= (A - aI)v_i + aIv_i \\ &= v_{i-1} + av_i. \end{aligned}$$

On the other hand,

$$PJ = \left[v_1 \middle| v_2 \middle| \dots \middle| v_k \right] J$$

and the $i-th$ column of PJ is Pj_i,

$$Pj_i = \left[v_1 \middle| v_2 \middle| \ldots \middle| v_k \right] j_i.$$

Remembering what the vector j_i is, and multiplying, we see that

$$Pj_i = v_{i-1} + av_i$$

as well.

Thus every column of AP is equal to the corresponding column of PJ, so

$$AP = PJ.$$

But Proposition 7 shows that the columns of P are linearly independent, so P is invertible. Multiplying on the right by P^{-1}, we see that

$$A = PJP^{-1}.$$

□

Example A.9. *Applying Theorem 8 to the matrix* $A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ *of examples 3 and 5, we see that*

$$\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix}^{-1}.$$

Now in general the Jordan Canonical Form of a matrix A will not consist of a single block, but will have a number of blocks, of varying sizes and associated to varying eigenvalues.

But in this situation we merely have to "assemble" the various blocks (to get the matrix) and the various chains of generalized eigenvectors (to get a basis). Actually, the word "merely" is a bit misleading, as the proof that we can to do is in fact a subtle one. Thus we shall not give the proof here. Instead, we shall merely illustrate the situation. In fact, in order to avoid complicated notation we shall merely illustrate the situation for 2-by-2 and 3-by-3 matrices.

Theorem A.10. *Let A be a 2-by-2 matrix. Then one of the following situations applies:*

(i) *A has distinct eigenvalues a and b. Let u be an eigenvector associated to the eigenvalue a and let v be an eigenvector associated to the eigenvalue b. Then* $A = PJP^{-1}$

with

$$J = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad \text{and} \quad P = \left[u \,\middle|\, v \right].$$

(Note in this case A is diagonalizable)

(ii) *A has a single eigenvalue a of multiplicity 2.*

(a) *A has two linearly independent eigenvectors u and v.*
Then $A = PJP^{-1}$ with

$$J = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \quad \text{and} \quad P = \left[u \,\middle|\, v \right].$$

(Note in this case A is diagonalizable. In fact, in this case $E_a = \mathbb{R}^2$ and A itself is the matrix $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.)

(b) *A has a single chain $\{ v_1, v_2 \}$ of generalized eigenvectors. Then $A = PJP^{-1}$ with*

$$J = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \quad \text{and} \quad P = \left[v_1 \,\middle|\, v_2 \right].$$

Theorem A.11. *Let A be a 3-by-3 matrix. Then one of the following situations applies:*

(i) *A has distinct eigenvalues a, b, and c. Let u be an eigenvector associated to the eigenvalue a, let v be an eigenvector associated to the eigenvalue b, and let w be an eigenvector associated to the eigenvalue c. Then $A = PJP^{-1}$ with*

$$J = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \quad \text{and} \quad P = \left[u \,\middle|\, v \,\middle|\, w \right].$$

(Note in this case A is diagonalizable.)

(ii) *A has an eigenvalue a of multiplicity two and an eigenvalue b of multiplicity one.*

(a) *A has two linearly independent eigenvectors u and v associated to the eigenvalue a. Let w be an eigenvector associated to the eigenvalue b. Then $A = PJP^{-1}$ with*

$$J = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \quad \text{and} \quad P = \left[u \,\middle|\, v \,\middle|\, w \right].$$

(Note in this case A is diagonalizable.)

(b) *A has a single chain $\{ u_1, u_2 \}$ of generalized eigenvectors associated to the eigenvalue a. Let v be an eigenvector associated to the eigenvalue b. Then $A = PJP^{-1}$*

with

$$J = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \quad \text{and} \quad P = \left[u_1 \,\middle|\, u_2 \,\middle|\, v \right].$$

(iii) *A has a single eigenvalue a of multiplicity three.*

(a) *A has three linearly independent eigenvectors u, v, and w. Then* $A = PJP^{-1}$ *with*

$$J = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \quad \text{and} \quad P = \left[u \,\middle|\, v \,\middle|\, w \right].$$

(Note in this case A is diagonalizable. In fact, in this case $E_a = \mathbb{R}^3$ *and A itself is the matrix* $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$*.)*

(b) *A has a chain* $\{ u_1, u_2 \}$ *of generalized eigenvectors and an eigenvector v with* $\{ u_1, u_2, v \}$ *linearly independent. Then* $A = PJP^{-1}$ *with*

$$J = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \quad \text{and} \quad P = \left[u_1 \,\middle|\, u_2 \,\middle|\, v \right].$$

(c) *A has a single chain* $\{ u_1, u_2, u_3 \}$ *of generalized eigenvectors. Then* $A = PJP^{-1}$ *with*

$$J = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \quad \text{and} \quad P = \left[u_1 \,\middle|\, u_2 \,\middle|\, u_3 \right].$$

Now we would like to apply Theorems 10 and 11. In order to do so, we need to have an effective method to determine which of the cases we are in, and we give that here (without proof).

Definition A.12. *Let λ be an eigenvalue of A. Then for any positive integer i,*

$$\begin{aligned} E^i_\lambda &= \{ v \mid (A - \lambda I)^i(v) = 0 \} \\ &= ker((A - \lambda I)^i). \end{aligned}$$

Note that E_λ^i *consists of generalized eigenvectors of index at most* i *(and the 0 vector), and is a subspace. Note also that*

$$E_\lambda = E_\lambda^1 \subseteq E_\lambda^2 \subseteq \ldots \subseteq \tilde{E}_\lambda.$$

In general, the Jordan canonical form of A is determined by the dimensions of all the spaces E_λ^i, but this determination can be a bit complicated. For eigenvalues of multiplicity at most 3, however, the situation is simpler –we need only consider the eigenspaces E_λ. This is a consequence of the following general result:

Proposition A.13. *Let* λ *be an eigenvalue of* A*. Then the number of blocks in the Jordan Canonical Form of* A *corresponding to* λ *is equal to* $\dim E_\lambda$*.*

Idea of Proof. Suppose there are k such blocks. Since each block corresponds to a chain of generalized eigenvectors, there are k such chains. Now the bottom of the chain is an (ordinary) eigenvector, so we get k eigenvectors in this way. It can be shown that these k eigenvectors are always linearly independent and that they always span E_λ, i.e., that they are a basis of E_λ. Thus E_λ has a basis consisting of k vectors, so $\dim E_\lambda = k$.

□

We can now determine the Jordan Canonical Forms of 1-by-1, 2-by-2, and 3-by-3 matrices, using the following consequences of this proposition.

Corollary A.14. *Let* λ *be an eigenvalue of* A *of multiplicity 1. Then* $\dim E_\lambda^1 = 1$ *and the submatrix of the Jordan Canonical Form of* A *corresponding to the eigenvalue* λ *is a single 1-by-1 block.*

Corollary A.15. *Let* λ *be an eigenvalue of* A *of multiplicity two. Then there are the following possibilities:*

(a) $\dim E_\lambda^1 = 2$*. In this case, the submatrix of the Jordan Canonical Form of* A *corresponding to the eigenvalue* λ *consists of two 1-by-1 blocks.*

(b) $\dim E_\lambda^1 = 1$, $\dim E_\lambda^2 = 2$*. In this case, the submatrix of the Jordan Canonical Form of* A *corresponding to the eigenvalue* λ *consists of a single 2-by-2 block.*

Corollary A.16. *Let λ be an eigenvalue of A of multiplicity three. Then there are the following possibilities:*

(a) $\dim E^1_\lambda = 3$. *In this case, the submatrix of the Jordan Canonical Form of A corresponding to the eigenvalue λ consists of three 1-by-1 blocks.*

(b) $\dim E^1_\lambda = 2$, $\dim E^2_\lambda = 3$. *In this case, the submatrix of the Jordan Canonical Form of A corresponding to the eigenvalue λ consists of a 2-by-2 block and a 1-by-1 block.*

(c) $\dim E^1_\lambda = 1$, $\dim E^2_\lambda = 2$, $\dim E^3_\lambda = 3$. *In this case, the submatrix of the Jordan Canonical Form of A corresponding to the eigenvalue λ consists of a single 3-by-3 block.*

Now we shall do several examples. We will only do examples where A is not diagonalizable.

Example A.17. $A= \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$.

A has characteristic polynomial $\det(\lambda I - A) = (\lambda+1)^2(\lambda-3)$. *Thus A has an eigenvalue –1 of multiplicity two and an eigenvalue 3 of multiplicity one. Computation shows that the eigenspace $E_{-1} = ker(A - (-I))$ has basis $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$, so $\dim E_{-1} = 1$ and we are in Corollary 15 case (b). Then we further compute that $E^2_{-1} = ker((A-(-I))^2)$ has basis $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, so is 2-dimensional, as we expect. More to the point, we may choose any generalized eigenvector of index 2, i.e., any vector in E^2_{-1} that is not in E^1_{-1}, as the top of a chain. We choose $u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and then we have $u_1 = (A - (-I))u_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, and $\{ u_1, u_2 \}$ form a chain.*

We also compute that, for the eigenvalue 3, the eigenspace E_3 has basis $\left\{ v = \begin{bmatrix} -5 \\ -6 \\ 1 \end{bmatrix} \right\}$. Hence we see that

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -5 \\ -2 & 0 & -6 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ -2 & 0 & -6 \\ -1 & 1 & 1 \end{bmatrix}^{-1}.$$

Example A.18. $A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix}$.

A has characteristic polynomial $\det(\lambda I - A) = (\lambda - 1)^3$*, so A has one eigenvalue 1 of multiplicity three. Computation shows that* $E_1 = \ker(A - I))$ *has basis* $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$*, so* $\dim E_1 = 2$ *and we are in Corollary 16 case (b). Computation then shows that* $\dim E_1^2 = 3$ *(i.e.,* $(A - I)^2 = 0$ *and* E_1^2 *is all of* $\mathbb{R}^3$*) with basis* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$*. We may choose* u_2 *to be any vector in* E_1^2 *that is not in* E_1^1*, and we shall choose* $u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$*. Then* $u_1 = (A - I)u_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$*, and* $\{ u_1, u_2 \}$ *form a chain. For the third vector v we may choose any vector in* E_1 *such that* $\{ u_1, v \}$ *is linearly independent. We choose* $v = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$*. Hence we see that*

$$\begin{bmatrix} 2 & 1 & 1 \\ -2 & -1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}.$$

Example A.19. $A = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$.

A has characteristic polynomial $\det(\lambda I - A) = (\lambda - 2)^3$*, so A has one eigenvalue 2 of multiplicity three. Computation shows that* $E_2 = \ker(A - 2I))$ *has basis* $\left\{ \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \right\}$*, so* $\dim E_2^1 = 1$ *and we are in Corollary 16 case (c). Then computation shows that* $E_2^2 = \ker(A - 2I)^2$ *has basis* $\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$*. (Note that* $\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = 3/2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + 1/2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$*.) Computation then shows that* $\dim E_2^3 = 3$ *(i.e.,* $(A - 2I)^3 = 0$ *and* E_2^3 *is all of* $\mathbb{R}^3$*) with* *basis* $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$. *We may choose* u_3 *to be any vector in* $\mathbb{R}^3$ *that is not in* E_2^2, *and we shall choose* $u_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$. *Then* $u_2 = (A-2I)u_3 = \begin{bmatrix}3\\1\\-7\end{bmatrix}$ *and* $u_1 = (A-2I)u_2 = \begin{bmatrix}2\\2\\-6\end{bmatrix}$, *and then* $\{ u_1, u_2, u_3 \}$ *form a chain. Hence we see that*

$$\begin{bmatrix}5 & 0 & 1\\ 1 & 1 & 0\\ -7 & 1 & 0\end{bmatrix} = \begin{bmatrix}2 & 3 & 1\\ 2 & 1 & 0\\ -6 & -7 & 0\end{bmatrix} \begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix} \begin{bmatrix}2 & 3 & 1\\ 2 & 1 & 0\\ -6 & -7 & 0\end{bmatrix}^{-1}.$$

<u>A. Exercises.</u> For each matrix A, write $A = PJP^{-1}$ with P an invertible matrix and J a matrix in Jordan canonical form.

1. $A = \begin{bmatrix} 75 & 56 \\ -90 & -67 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 3)(\lambda - 5).$

2. $A = \begin{bmatrix} -50 & 99 \\ -20 & 39 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda + 6)(\lambda + 5).$

3. $A = \begin{bmatrix} -18 & 9 \\ -49 & 24 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 3)^2.$

4. $A = \begin{bmatrix} 1 & 1 \\ -16 & 9 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 5)^2.$

5. $A = \begin{bmatrix} 2 & 1 \\ -25 & 12 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 7)^2.$

6. $A = \begin{bmatrix} -15 & 9 \\ -25 & 15 \end{bmatrix}, \quad \det(\lambda I - A) = \lambda^2.$

7. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda + 1)(\lambda - 1)(\lambda - 3).$

8. $A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 1)(\lambda - 2)(\lambda - 4).$

9. $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda + 3)^2(\lambda - 1).$

10. $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 3)^2(\lambda - 8).$

11. $A = \begin{bmatrix} 5 & 2 & 1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 4)^2(\lambda - 2).$

12. $A = \begin{bmatrix} 8 & -3 & -3 \\ 4 & 0 & -2 \\ -2 & 1 & 3 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 2)^2(\lambda - 7).$

13. $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda + 2)^2(\lambda - 4).$

14. $A = \begin{bmatrix} 3 & 0 & 0 \\ 9 & -5 & -18 \\ -4 & 4 & 12 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 3)^2(\lambda - 4).$

15. $A = \begin{bmatrix} -6 & 9 & 0 \\ -6 & 6 & -2 \\ 9 & -9 & 3 \end{bmatrix}, \quad \det(\lambda I - A) = \lambda^2(\lambda - 3).$

16. $A = \begin{bmatrix} -18 & 42 & 168 \\ 1 & -7 & -40 \\ -2 & 6 & 27 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 3)^2(\lambda + 4).$

17. $A = \begin{bmatrix} -1 & 1 & -1 \\ -10 & 6 & -5 \\ -6 & 3 & -2 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 1)^3.$

18. $A = \begin{bmatrix} 0 & -4 & 1 \\ 2 & -6 & 1 \\ 4 & -8 & 0 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda + 2)^3.$

19. $A = \begin{bmatrix} -4 & 1 & 2 \\ -5 & 1 & 3 \\ -7 & 2 & 3 \end{bmatrix}, \quad \det(\lambda I - A) = \lambda^3.$

20. $A = \begin{bmatrix} -4 & -2 & 5 \\ -1 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda + 1)^3.$

A. Answers to odd-numbered exercises

1. $A = \begin{bmatrix} -7 & 4 \\ 9 & -5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -7 & 4 \\ 9 & -5 \end{bmatrix}^{-1}.$

3. $A = \begin{bmatrix} -21 & 1 \\ -49 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -21 & 1 \\ -49 & 0 \end{bmatrix}^{-1}.$

5. $A = \begin{bmatrix} -5 & 1 \\ -25 & 0 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -5 & 1 \\ -25 & 0 \end{bmatrix}^{-1}.$

7. $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}^{-1}.$

9. $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}.$

11. $A = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}.$

13. $A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}.$

15. $A = \begin{bmatrix} -3 & -1 & -2 \\ -2 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & -1 & -2 \\ -2 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}^{-1}.$

17. $A = \begin{bmatrix} -2 & 1 & 1 \\ -10 & 0 & 2 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ -10 & 0 & 2 \\ -6 & 0 & 0 \end{bmatrix}^{-1}.$

19. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 3 & 1 \end{bmatrix}^{-1}.$

B. Homogeneous Systems with Constant Coefficients:

The Nondiagonalizable Case

We will now see how to use Jordan Canonical Form to solve systems $Y' = AY$ where the coefficient matrix A is not diagonalizable.

The key to understanding systems is to investigate a system $Z' = JZ$ where J is a matrix consisting of a single Jordan block.

Theorem B.1. *Let J be a k-by-k Jordan block with eigenvalue a,*

$$J = \begin{bmatrix} a & 1 & & & & \\ & a & 1 & & 0 & \\ & & a & 1 & & \\ & & & \ddots & \ddots & \\ & 0 & & & a & 1 \\ & & & & & a \end{bmatrix}.$$

Then the system $Z' = JZ$ has the solution

$$Z = e^{ax} \begin{bmatrix} 1 & x & x^2/2! & x^3/3! & \cdots & x^{k-1}/(k-1)! \\ & 1 & x & x^2/2! & \cdots & x^{k-2}/(k-2)! \\ & & 1 & x & \cdots & x^{k-3}/(k-3)! \\ & & & \ddots & & \vdots \\ & & & & & x \\ & & & & & 1 \end{bmatrix} C$$

where $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ *is a vector of arbitrary constants $c_1, c_2, \ldots, c_k$.*

Proof. We will prove this in the cases $k = 1$, 2, and 3, which illustrate the pattern. As you will see, the proof is a simple application of the standard technique for solving first-order linear differential equations.

The case $k = 1$: Here we are considering the system

$$[z_1'] = [a][z_1]$$

which is nothing other than the differential equation

$$z_1' = az_1.$$

The solution to this differential equation is very familiar. It is

$$z_1 = c_1 e^{ax}$$

which we can certainly write as

$$[z_1] = e^{az}[1][c_1].$$

The case $k = 2$: Here we are considering the system

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

which is nothing other than the pair of differential equations

$$\begin{aligned} z_1' &= az_1 + z_2 \\ z_2' &= \quad az_2. \end{aligned}$$

We recognize the second equation as having the solution

$$z_2 = c_2 e^{ax}$$

and we substitute this into the first equation to get

$$z_1' = az_1 + c_2 e^{ax}.$$

To solve this, we rewrite this as

$$z_1' - az_1 = c_2 e^{ax}$$

and recognize that this differential equation has integrating factor e^{-ax}. Multiplying by this factor, we find

$$\begin{aligned} e^{-ax}(z_1' - az_1) &= c_2 \\ (e^{-ax} z_1)' &= c_2 \\ e^{-ax} z_1 &= \int c_2 \, dx = c_1 + c_2 x \end{aligned}$$

so

$$z_1 = e^{ax}(c_1 + c_2 x).$$

Thus our solution is

$$\begin{aligned} z_1 &= e^{ax}(c_1 + c_2 x) \\ z_2 &= \quad e^{ax} c_2 \end{aligned}$$

which we see we can rewrite as

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = e^{ax} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The case $k = 3$: Here we are considering the system

$$\begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

which is nothing other than the triple of differential equations

$$\begin{aligned} z_1' &= az_1 + z_2 \\ z_2' &= az_2 + z_3 \\ z_3' &= az_3. \end{aligned}$$

If we just concentrate on the last two equations, we see we are in the $k = 2$ case. Referring to that case, we see that our solution is

$$\begin{aligned} z_2 &= e^{ax}(c_2 + c_3 x) \\ z_3 &= e^{ax} c_3. \end{aligned}$$

Substituting the value of z_2 into the equation for z_1, we obtain

$$z_1' = az_1 + e^{ax}(c_2 + c_3 x).$$

To solve this, we rewrite this as

$$z_1' - az_1 = e^{ax}(c_2 + c_3 x)$$

and recognize that this differential equation has integrating factor e^{-ax}. Multiplying by this factor, we find

$$\begin{aligned} e^{-ax}(z_1' - az_1) &= c_2 + c_3 x \\ (e^{-ax} z_1)' &= c_2 + c_3 x \\ e^{-ax} z_1 &= \int (c_2 + c_3 x)\, dx = c_1 + c_2 x + c_3 (x^2/2) \end{aligned}$$

so

$$z_1 = e^{ax}(c_1 + c_2 x + c_3 (x^2/2)).$$

Thus our solution is

$$\begin{aligned} z_1 &= e^{ax}(c_1 + c_2 x + c_3(x^2/2)) \\ z_2 &= e^{ax}(c_2 + c_3 x) \\ z_3 &= e^{ax} c_3 \end{aligned}$$

which we see we can rewrite as

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = e^{ax} \begin{bmatrix} 1 & x & x^2/2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

□

To apply Theorem 1 in general we make the following observation.

Remark B.2. Suppose that $Z' = JZ$ where J is a matrix in Jordan canonical form, but one consisting of several blocks, not just one block. We can see that this systems decomposes into several systems, one corresponding to each block, and that these systems are "uncoupled" (i.e., have nothing to do with each other), so we may solve them each separately, using Theorem 1, and then simply assemble these individual solutions together to obtain a solution of the general system.

Now consider a matrix system

$$Y' = AY.$$

Our method of solution is entirely analogous to that in the diagonalizable case.

Step 1. Write $A = PJP^{-1}$ with J in Jordan Canonical form, so the system becomes

$$\begin{aligned} Y' &= (PJP^{-1})Y \\ Y' &= PJ(P^{-1}Y) \\ P^{-1}Y' &= J(P^{-1}Y) \\ (P^{-1}Y)' &= J(P^{-1}Y) \end{aligned}$$

Step 2. Set $Z = P^{-1}Y$, so this system becomes

$$Z' = JZ$$

and use Theorem 1 and Remark 2 to solve this system for Z.

Step 3. Since $Z = P^{-1}Y$, we have that

$$Y = PZ$$

is the solution to our original system.

We now illustrate this (again confining our illustrations to the case that A is not diagonalizable).

Example B.3. *Consider the system*

$$Y' = AY \quad where \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

We saw in section A example 9 that $A = PJP^{-1}$ *with*

$$P = \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix} \quad and \quad J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then $Z' = JZ$ *has solution*

$$Z = e^{2x} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{2x} & xe^{2x} \\ 0 & e^{2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1e^{2x} + c_2xe^{2x} \\ c_2e^{2x} \end{bmatrix}$$

and so $Y = PZ$, *i.e.,*

$$\begin{aligned} Y &= \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix} e^{2x} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} e^{2x} & xe^{2x} \\ 0 & e^{2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} -2e^{2x} & -2xe^{2x} + e^{2x} \\ -4e^{2x} & -4xe^{2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} (-2c_1 + c_2)e^{2x} - 2c_2xe^{2x} \\ -4c_1e^{2x} - 4c_2xe^{2x} \end{bmatrix}. \end{aligned}$$

Example B.4. *Consider the system*

$$Y' = AY \quad where \quad A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}.$$

We saw in section A example 17 that $A = PJP^{-1}$ *with*

$$P = \begin{bmatrix} 1 & 0 & -5 \\ -2 & 0 & -6 \\ -1 & 1 & 1 \end{bmatrix} \quad and \quad J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then $Z' = JZ$ *has solution*

$$Z = \begin{bmatrix} e^{-x} & xe^{-x} & 0 \\ 0 & e^{-x} & 0 \\ 0 & 0 & e^{3x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and so $Y = PZ$*, i.e.,*

$$\begin{aligned} Y &= \begin{bmatrix} 1 & 0 & -5 \\ -2 & 0 & -6 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-x} & xe^{-x} & 0 \\ 0 & e^{-x} & 0 \\ 0 & 0 & e^{3x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{-x} & xe^{-x} & -5e^{3x} \\ -2e^{-x} & -2xe^{-x} & -6e^{3x} \\ -e^{-x} & -xe^{-x} + e^{-x} & e^{3x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} c_1 e^{-x} + c_2 xe^{-x} - 5c_3 e^{3x} \\ -2c_1 e^{-x} - 2c_2 xe^{-x} - 6c_3 e^{3x} \\ (-c_1 + c_2)e^{-x} - c_2 xe^{-x} + c_3 e^{3x} \end{bmatrix}. \end{aligned}$$

Example B.5. *Consider the system*

$$Y' = AY \quad where \quad A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix}.$$

We saw in section A example 18 that $A = PJP^{-1}$ *with*

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad and \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $Z' = JZ$ *has solution*

$$Z = \begin{bmatrix} e^{x} & xe^{x} & 0 \\ 0 & e^{x} & 0 \\ 0 & 0 & e^{x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and so $Y = PZ$*, i.e.,*

$$
\begin{aligned}
Y &= \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^x & xe^x & 0 \\ 0 & e^x & 0 \\ 0 & 0 & e^x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^x & xe^x + e^x & e^x \\ -2e^x & -2xe^x & 0 \\ e^x & xe^x & e^x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (c_1 + c_2 + c_3)e^x + c_2 xe^x \\ -2c_1 e^x \qquad - 2c_2 xe^x \\ (c_1 + c_3)e^x \qquad + c_2 xe^x \end{bmatrix}.
\end{aligned}
$$

Example B.6. *Consider the system*

$$
Y' = AY \quad where \quad A = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}.
$$

We saw in section A example 19 that $A = PJP^{-1}$ *with*

$$
P = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \\ -6 & -7 & 0 \end{bmatrix} \quad and \quad J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.
$$

Then $Z' = JZ$ *has solution*

$$
Z = \begin{bmatrix} e^{2x} & xe^{2x} & (x^2/2)e^{2x} \\ 0 & e^{2x} & xe^{2x} \\ 0 & 0 & e^{2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
$$

and so $Y = PZ$, *i.e.,*

$$
\begin{aligned}
Y &= \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \\ -6 & -7 & 0 \end{bmatrix} \begin{bmatrix} e^{2x} & xe^{2x} & (x^2/2)e^{2x} \\ 0 & e^{2x} & xe^{2x} \\ 0 & 0 & e^{2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} 2e^{2x} & 2xe^{2x} + 3e^{2x} & x^2e^{2x} + 3xe^{2x} + e^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} & x^2e^{2x} + xe^{2x} \\ -6e^{2x} & -6xe^{2x} - 7e2x & -3x^2e^{2x} - 7xe^{2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (2c_1 + 3c_2 + c_3)e^{2x} + (2c_2 + 3c_3)xe^{2x} \; + c_3x^2e^{2x} \\ (2c_1 + c_2)e^{2x} \qquad + (2c_2 + c_3)xe^{2x} \quad + c_3x^2e^{2x} \\ (-6c_1 - 7c_2)e^{2x} \quad + (-6c_2 - 7c_3)xe^{2x} - 3c_3x^2e^{2x} \end{bmatrix}.
\end{aligned}
$$

We conclude this section by showing how to solve initial value problems. This is just one more step, given what we have already done.

Example B.7. *Consider the initial value problem*

$$Y' = AY \quad where \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}, \quad and \quad Y(0) = \begin{bmatrix} 3 \\ -8 \end{bmatrix}.$$

In example 3 we saw that this system has the general solution

$$Y = \begin{bmatrix} (-2c_1 + c_2)e^{2x} - 2c_2xe^{2x} \\ -4c_1e^{2x} \qquad - 4c_2xe^{2x} \end{bmatrix}.$$

Applying the initial condition (i.e., substituting $x = 0$ in this matrix), gives

$$\begin{bmatrix} 3 \\ -8 \end{bmatrix} = Y(0) = \begin{bmatrix} -2c_1 + c_2 \\ -4c_1 \end{bmatrix}$$

with solution

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

Substituting these values in the above matrix gives

$$Y = \begin{bmatrix} 3e^{2x} - 14xe^{2x} \\ -8e^{2x} - 28te^{2x} \end{bmatrix}.$$

Example B.8. *Consider the initial value problem*

$$Y' = AY \quad where \quad A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}, \quad and \quad Y(0) = \begin{bmatrix} 8 \\ 32 \\ 5 \end{bmatrix}.$$

In example 4 we saw that this system has the general solution

$$Y = \begin{bmatrix} c_1e^{-x} \qquad + c_2xe^{-x} \quad - 5c_3xe^{3x} \\ -2c_1e^{-x} \quad - 2c_2xe^{-x} - 6c_3e^{3x} \\ (-c_1 + c_2)e^{-x} - c_2xe^{-x} + c_3e^{3x} \end{bmatrix}.$$

Applying the initial condition (i.e., substituting $t = 0$ in this matrix), gives

$$\begin{bmatrix} 8 \\ 32 \\ 5 \end{bmatrix} = Y(0) = \begin{bmatrix} c_1 \qquad - 5c_3 \\ -2c_1 \quad - 6c_3 \\ -c_1 + c_2 + c_3 \end{bmatrix}$$

with solution

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \\ -3 \end{bmatrix}.$$

Substituting these values in the above matrix gives

$$Y = \begin{bmatrix} -7e^{-x} + xe^{-x} + 15e^{3x} \\ 14e^{-x} - 2xe^{-x} + 18e^{3x} \\ 8e^{-x} - xe^{-x} - 3e^{3x} \end{bmatrix}.$$

Remark B.9. There is an alternate way of solving initial value problems. It is actually less effective than the method we have given for solving a single initial value problem, but has the advantage of expressing the solution directly in terms of the initial conditions. This makes it more effective if the same system $Y' = AY$ is to be solved for a variety of initial conditions.

Let us write our solution of $Z' = JZ$ as

$$Z(x) = M(x)C$$

where C is a vector of constants. The key observation is that $M(0) = I$, the identity matrix. Thus, if we wish to solve the initial value problem

$$Z' = JZ, \quad Z(0) = Z_0$$

we find that, in general,

$$Z(x) = M(x)C$$

and in particular

$$Z_0 = Z(0) = M(0)C = IC = C$$

so the solution to this initial value problem is

$$Z(x) = M(x)Z_0.$$

Now suppose we wish to solve the initial value problem

$$Y' = AY, \quad Y(0) = Y_0.$$

Then, if $A = PJP^{-1}$, we set $Z = P^{-1}Y$, just as before, to obtain the equation $Z' = JZ$, with solution $Z(x) = M(x)Z_0$. But $Z = P^{-1}Y$, i.e., $Z(x) = P^{-1}Y(x)$, and in particular $Z_0 = Z(0) = P^{-1}Y(0) = P^{-1}Y_0$. Substituting, we find

$$P^{-1}Y(x) = M(x)(P^{-1}Y_0)$$

or

$$Y(x) = (PM(x)P^{-1})Y_0.$$

.

Let us now use this technique.

Example B.10. *Consider the initial value problem*

$$Y' = AY \quad where \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}, \quad and \quad Y(0) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

As we have seen in example 3, $A = PJP^{-1}$ *with* $P = \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix}$ *and* $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. *Then* $M(x) = \begin{bmatrix} e^{2x} & xe^{2x} \\ 0 & e^{2x} \end{bmatrix}$ *and*

$$\begin{aligned} PM(x)P^{-1} &= \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} e^{2x} & xe^{2x} \\ 0 & e^{2x} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} e^{2x} - 2xe^{2x} & xe^{2x} \\ -4xe^{2x} & e^{2x} + 2xe^{2x} \end{bmatrix} \end{aligned}$$

so

$$\begin{aligned} Y(x) &= \begin{bmatrix} e^{2x} - 2xe^{2x} & xe^{2x} \\ -4xe^{2x} & e^{2x} + 2xe^{2x} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1e^{2x} + (-2a_1 + a_2)xe^{2x} \\ a_2e^{2x} + (-4a_1 + 2a_2)xe^{2x} \end{bmatrix}. \end{aligned}$$

In particular, if $Y(0) = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$, *then* $Y(x) = \begin{bmatrix} 3e^{2x} - 14xe^{2x} \\ -8e^{2x} - 28xe^{2x} \end{bmatrix}$, *recovering the result of example 8. But also, if* $Y(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, *then* $Y(x) = \begin{bmatrix} 2e^{2x} + xe^{2x} \\ 5e^{2x} + 2te^{2x} \end{bmatrix}$, *and if* $Y(0) = \begin{bmatrix} -4 \\ 15 \end{bmatrix}$, *then* $Y(x) = \begin{bmatrix} -4e^{2x} + 23xe^{2x} \\ 15e^{2x} + 46xe^{2x} \end{bmatrix}$, *etc.*

Remark B.11. Mathematicians have defined the matrix exponential and, with this definition, $M(x) = e^{Jx}$ and $PM(x)P^{-1} = e^{Ax}$.

<u>B. Exercises.</u> For each exercise, see the corresponding exercise in the section on Jordan Canonical form. In each problem:

a) Solve the system $Y' = AY$.

b) Solve the initial value problem $Y' = AY$, $Y(0) = Y_0$.

1. $A = \begin{bmatrix} 75 & 56 \\ -90 & -67 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2. $A = \begin{bmatrix} -50 & 99 \\ -20 & 39 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

3. $A = \begin{bmatrix} -18 & 9 \\ -49 & 24 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 41 \\ 98 \end{bmatrix}$.

4. $A = \begin{bmatrix} 1 & 1 \\ -16 & 9 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$.

5. $A = \begin{bmatrix} 2 & 1 \\ -25 & 12 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} -10 \\ -75 \end{bmatrix}$.

6. $A = \begin{bmatrix} -15 & 9 \\ -25 & 15 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 50 \\ 100 \end{bmatrix}$.

7. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 6 \\ -10 \\ 10 \end{bmatrix}$.

8. $A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$.

9. $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

10. $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

11. $A = \begin{bmatrix} 5 & 2 & 1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} -3 \\ 2 \\ 9 \end{bmatrix}$.

12. $A = \begin{bmatrix} 8 & -3 & -3 \\ 4 & 0 & -2 \\ -2 & 1 & 3 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 5 \\ 8 \\ 7 \end{bmatrix}$.

13. $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$.

14. $A = \begin{bmatrix} 3 & 0 & 0 \\ 9 & -5 & -18 \\ -4 & 4 & 12 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

15. $A = \begin{bmatrix} -6 & 9 & 0 \\ -6 & 6 & -2 \\ 9 & -9 & 3 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$.

16. $A = \begin{bmatrix} -18 & 42 & 168 \\ 1 & -7 & -40 \\ -2 & 6 & 27 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}$.

17. $A = \begin{bmatrix} -1 & 1 & -1 \\ -10 & 6 & -5 \\ -6 & 3 & 2 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 3 \\ 10 \\ 18 \end{bmatrix}$.

18. $A = \begin{bmatrix} 0 & -4 & 1 \\ 2 & -6 & 1 \\ 4 & -8 & 0 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.

19. $A = \begin{bmatrix} -4 & 1 & 2 \\ -5 & 1 & 3 \\ -7 & 2 & 3 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 6 \\ 11 \\ 9 \end{bmatrix}$.

20. $A = \begin{bmatrix} -4 & -2 & 5 \\ -1 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix}$ and $Y_0 = \begin{bmatrix} 9 \\ 5 \\ 8 \end{bmatrix}$.

B. Answers to odd-numbered exercises.

1a. $Y = \begin{bmatrix} -7c_1e^{3x} + 4c_2e^{5x} \\ 9c_1e^{3x} - 5c_2e^{5x} \end{bmatrix}$ b. $Y = \begin{bmatrix} -7e^{3x} + 8e^{5x} \\ 9e^{3x} - 10e^{5x} \end{bmatrix}$

3a. $Y = \begin{bmatrix} (-21c_1 + c_2)e^{3x} - 21c_2xe^{3x} \\ -49c_1e^{3x} \quad - 49c_2xe^{3x} \end{bmatrix}$ b. $Y = \begin{bmatrix} 41e^{3x} + 21xe^{3x} \\ 98e^{3x} + 49xe^{3x} \end{bmatrix}$

5a. $Y = \begin{bmatrix} (-5c_1 + c_2)e^{7x} - 5c_2xe^{7x} \\ -25c_1e^{7x} \quad - 25c_2xe^{7x} \end{bmatrix}$ b. $Y = \begin{bmatrix} -10e^{7x} - 25te^{7x} \\ -75e^{7x} - 125te^{7x} \end{bmatrix}$

7a. $Y = \begin{bmatrix} 2c_2e^{x} \\ c_1e^{-x} + c_2e^{x} - 3c_3e^{3x} \\ c_1e^{-x} + c_2e^{x} + c_3e^{3x} \end{bmatrix}$ b. $Y = \begin{bmatrix} 6e^{x} \\ 2e^{-x} + 3e^{x} - 15e^{3x} \\ 2e^{-x} + 3e^{x} + 5e^{3x} \end{bmatrix}$

9a. $Y = \begin{bmatrix} (-c_1 - 2c_2)e^{-3x} - 2c_3e^{x} \\ c_1e^{-3x} - c_3e^{x} \\ c_2e^{-3x} + c_3e^{x} \end{bmatrix}$ b. $Y = \begin{bmatrix} 0 \\ 2e^{-3x} \\ -e^{-3x} \end{bmatrix}$

11a. $Y = \begin{bmatrix} (-c_1 - 2c_2)e^{4x} - c_3e^{2x} \\ c_2e^{4x} + c_3e^{2x} \\ c_1e^{4x} + c_3e^{2x} \end{bmatrix}$ b. $Y = \begin{bmatrix} 2e^{4x} - 5e^{2x} \\ -3e^{43x} + 5e^{2x} \\ 4e^{4x} + 5e^{2x} \end{bmatrix}$

13a. $Y = \begin{bmatrix} -c_1e^{-2x} - c_2xe^{-2x} \\ -c_1e^{-2x} - c_2xe^{-2x} + c_3e^{4x} \\ c_2e^{-2x} \quad + c_3e^{4x} \end{bmatrix}$ b. $Y = \begin{bmatrix} -e^{-2x} - 2xe^{-2x} \\ -e^{-2x} - 2xe^{-2x} + 4e^{4x} \\ 2e^{-2x} \quad + 4e^{4x} \end{bmatrix}$

15a. $Y = \begin{bmatrix} (-3c_1 - c_2) - 3c_2x - 2c_3e^{3x} \\ (-2c_1 - c_2) - 2c_2x - 2c_3e^{3x} \\ (3c_1 + c_2) + 3c_2x + 3c_3e^{3x} \end{bmatrix}$ b. $Y = \begin{bmatrix} -9 - 9x + 10e^{3x} \\ -7 - 6x + 10e^{3x} \\ 9 + 9x - 15e^{3x} \end{bmatrix}$

17a. $Y = \begin{bmatrix} (-2c_1 + c_2 + c_3)e^{x} - 2c_2xe^{x} \\ (-10c_1 + 2c_3)e^{x} - 10c_2xe^{x} \\ -6c_1e^{x} - 6c_2xe^{x} \end{bmatrix}$ b. $Y = \begin{bmatrix} 3e^{x} - 14xe^{x} \\ 10e^{x} - 70xe^{x} \\ 18e^{x} - 42xe^{x} \end{bmatrix}$

19a. $Y = \begin{bmatrix} (c_1 + 2c_2) \quad + (c_2 + 2c_3)x + (c_3/2)x^2 \\ (2c_1 + 3c_2) \quad + (2c_2 + 3c_3)x + c_3x^2 \\ (c_1 + 3c_2 + c_3) + (c_2 + 3c_3)t + (c_3/2)x^2 \end{bmatrix}$ b. $Y = \begin{bmatrix} 6 + 5x + x^2 \\ 11 + 8x + 2x^2 \\ 9 + 7x + x^2 \end{bmatrix}$

C. The Matrix Exponential

In this section we will discuss the matrix exponential and its use in solving systems $Y' = AY$.

Our first task is to ask what it means to take a matrix exponential. To answer this, we are guided by ordinary exponentials. Recall that, for any complex number z, the exponential e^z is given by

$$e^z = 1 + z + z^2/2! + z^3/3! + z^4/4! + \dots \qquad .$$

With this in mind, we define the matrix exponential as follows.

Definition C.1. *Let T be a square matrix. Then e^T is defined by*

$$e^T = I + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \frac{1}{4!}T^4 + \dots \qquad .$$

(For this definition to make sense we need to know that this series always converges, and it does.)

Recall that the differential equation $y' = ay$ has the solution $y = ce^{ax}$. The situation for $Y' = AY$ is very analogous. (Note that we use Γ rather than C to denote a vector of constants for reasons that will become clear a little later. Note that Γ is on the right in theorem 2 below, a consequence of the fact that matrix multiplication is not commutative.)

Theorem C.2.

(1) *Let A be a square matrix. Then the general solution of*

$$Y' = AY$$

is given by

$$Y = e^{Ax}\Gamma$$

where Γ is a vector of arbitrary constants.

(2) *The initial value problem*

$$Y' = AY, \qquad Y(0) = Y_0$$

has solution

$$Y = e^{Ax}Y_0.$$

Outline of proof. We first compute e^{Ax}. In order to do so, note that $(Ax)^2 = (Ax)(Ax) = (AA)(xx) = A^2x^2$ as matrix multiplication commutes with scalar multiplication, and $(Ax)^3 = (Ax)^2(Ax) = (A^2x^2)(Ax) = (A^2A)(x^2x) = A^3x^3$, and similarly $(Ax)^k = A^kx^k$ for any k. Then, substituting in definition 1, we have that

$$Y = e^{Ax}\Gamma = (I + Ax + \frac{1}{2!}A^2x^2 + \frac{1}{3!}A^3x^3 + \frac{1}{4!}A^4x^4 + \ldots)\Gamma.$$

To find Y', we may differentiate this series term-by-term. (This claim requires proof, but we shall not give it here.) Remembering that A and Γ are constant matrices, we see that

$$\begin{aligned} Y' &= (A + \frac{1}{2!}A^2(2x) + \frac{1}{3!}A^3(3x^2) + \frac{1}{4!}A^4(4x^3) + \ldots)\Gamma \\ &= (A + A^2x + \frac{1}{2!}A^3x^2 + \frac{1}{3!}A^4x^3 + \ldots)\Gamma \\ &= A(I + Ax + \frac{1}{2!}A^2x^2 + \frac{1}{3!}A^3x^3 + \ldots)\Gamma \\ &= A(e^{Ax}\Gamma) = AY \end{aligned}$$

as claimed.

(2) By part 1 we know that $Y' = AY$ has solution $Y = e^{Ax}\Gamma$. We use the initial condition to solve for Γ. Setting $x = 0$, we have

$$Y_0 = Y(0) = \qquad e^{A0}\Gamma = e^0\Gamma = I\Gamma = \Gamma$$

(where e^0 means the exponential of the zero matrix, and the value of this is the identity matrix), so $\Gamma = Y_0$ and $Y = e^{Ax}\Gamma = e^{Ax}Y_0$. □

In the remainder of this section we shall see how to translate the theoretical solution of $Y' = AY$ given by theorem 2 into a practical one. To keep our notation simple, we will stick to 2-by-2 or 3-by-3 cases, but the principle is the same regardless of the size of the matrix.

One case is relatively easy.

Lemma C.3. *If D is a diagonal matrix,*

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & 0 & \\ & 0 & \ddots & \\ & & & d_n \end{bmatrix}$$

then e^{Dx} is the diagonal matrix

$$e^{Dx} = \begin{bmatrix} e^{d_1 x} & & & \\ & e^{d_2 x} & 0 & \\ & 0 & \ddots & \\ & & & e^{d_n x} \end{bmatrix}.$$

Proof. Suppose, for simplicity, that D is 2-by-2,

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}.$$

Then you can easily compute that $D^2 = \begin{bmatrix} {d_1}^2 & 0 \\ 0 & {d_2}^2 \end{bmatrix}$, $D^3 = \begin{bmatrix} {d_1}^3 & 0 \\ 0 & {d_2}^3 \end{bmatrix}$, and similarly $D^k = \begin{bmatrix} {d_1}^k & 0 \\ 0 & {d_2}^k \end{bmatrix}$ for any k.

Then, as in the proof of theorem 2,

$$\begin{aligned} e^{Dx} &= I + Dx + \frac{1}{2!}D^2x^2 + \frac{1}{3!}D^3x^3 + \frac{1}{4!}D^4x^4 + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} x + \frac{1}{2!} \begin{bmatrix} {d_1}^2 & 0 \\ 0 & {d_2}^2 \end{bmatrix} x^2 + \frac{1}{3!} \begin{bmatrix} {d_1}^3 & 0 \\ 0 & {d_2}^3 \end{bmatrix} x^3 + \dots \\ &= \begin{bmatrix} 1 + d_1x + \frac{1}{2!}(d_1x)^2 + \frac{1}{3!}(d_1x)^3 + \dots & 0 \\ 0 & 1 + d_2x + \frac{1}{2!}(d_2x)^2 + \frac{1}{3!}(d_2x)^3 + \dots \end{bmatrix} \end{aligned}$$

which we recognize as

$$= \begin{bmatrix} e^{d_1 x} & 0 \\ 0 & e^{d_2 x} \end{bmatrix}.$$

□

Example C.4. *We wish to find the general solution of $Y' = DY$ where*

$$D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

To do so we directly apply theorem 2 and lemma 3. The solution is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Y = e^{Dx}\Gamma = \begin{bmatrix} e^{4x} & 0 \\ 0 & e^{-x} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \gamma_1 e^{4x} \\ \gamma_2 e^{-x} \end{bmatrix}.$$

Now suppose we want to find the general solution of $Y' = AY$ where $A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$. We may still apply theorem 2 to conclude that the solution is $Y = e^{Ax}\Gamma$. We again try to calculate e^{Ax}. Now we find

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 7 & -9 \\ -6 & 10 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 25 & -39 \\ -26 & 38 \end{bmatrix}, \quad \dots$$

so

$$e^{Ax} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} x + \frac{1}{2!} \begin{bmatrix} 7 & -9 \\ -6 & 10 \end{bmatrix} x^2 + \frac{1}{3!} \begin{bmatrix} 25 & -39 \\ -26 & 38 \end{bmatrix} x^3 + \dots$$

which looks like a hopeless mess. But in fact, the situation is not so hard!

Lemma C.5. *Let S and T be two matrices and suppose*

$$S = PTP^{-1}$$

for some invertible matrix P. Then

$$S^k = PT^kP^{-1} \qquad \text{for every } k,$$

and

$$e^S = Pe^TP^{-1}.$$

Proof. We simply compute

$$\begin{aligned} S^2 &= SS = (PTP^{-1})(PTP^{-1}) = PT(P^{-1}P)TP^{-1} = PTITP^{-1} \\ &= PTTP^{-1} = PT^2P^{-1}, \\ S^3 &= S^2S = (PT^2P^{-1})(PTP^{-1}) = PT^2(P^{-1}P)TP^{-1} = PT^2ITP^{-1} \\ &= PT^2TP^{-1} = PT^3P^{-1}, \\ S^4 &= S^3S = (PT^3P^{-1})(PTP^{-1}) = PT^3(P^{-1}P)TP^{-1} = PT^3ITP^{-1} \\ &= PT^3TP^{-1} = PT^4P^{-1}, \end{aligned}$$

etc.

Then

$$\begin{aligned} e^S &= I + S + \frac{1}{2!}S^2 + \frac{1}{3!}S^3 + \frac{1}{4!}S^4 + \dots \\ &= PIP^{-1} + PTP^{-1} + \frac{1}{2!}PT^2P^{-1} + \frac{1}{3!}PT^3P^{-1} + \frac{1}{4!}PT^4P^{-1} + \dots \\ &= P(I + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \frac{1}{4!}T^4 + \dots)P^{-1} \\ &= Pe^TP^{-1} \end{aligned}$$

as claimed. ∎

With this in hand let us return to our problem.

Example C.6. *We wish to find the general solution of $Y' = AY$ where*

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

The matrix A has eigenvalues 4 and -1. The eigenspace E_4 has basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and the eigenspace E_{-1} has basis $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$, so we see that

$$A = PDP^{-1}$$

where $P = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$. Then $Ax = (PDP^{-1})x = P(Dx)P^{-1}$ (as scalar multiplication commutes with matrix multiplication) so by lemma 5 and example 4

$$\begin{aligned} e^{Ax} &= Pe^{Dx}P^{-1} \\ &= \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{4x} & 0 \\ 0 & e^{-x} \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{2}{5}e^{4x} + \frac{3}{5}e^{-x} & -\frac{3}{5}e^{4x} + \frac{3}{5}e^{-x} \\ -\frac{2}{5}e^{4x} + \frac{2}{5}e^{-x} & \frac{3}{5}e^{4x} + \frac{2}{5}e^{-x} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} Y = e^{Ax}\Gamma &= e^{Ax} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} (\frac{2}{5}\gamma_1 - \frac{3}{5}\gamma_2)e^{4x} + (\frac{3}{5}\gamma_1 + \frac{3}{5}\gamma_2)e^{-x} \\ (-\frac{2}{5}\gamma_1 + \frac{3}{5}\gamma_2)e^{4x} + (\frac{2}{5}\gamma_1 + \frac{2}{5}\gamma_2)e^{-x} \end{bmatrix}. \end{aligned}$$

Example C.7. *We wish to find the general solution of $Y' = AY$ where*

$$A = \begin{bmatrix} 2 & -3 & -3 \\ 2 & -2 & -2 \\ -2 & 1 & 1 \end{bmatrix}.$$

The matrix A has eigenvalues -1, 0, 2 *with* E_{-1} *having basis* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, E_0 *having basis* $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ *and* E_2 *having basis* $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$, *so we see that*

$$A = PDP^{-1}$$

where $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ *and* $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. *Then*

$$\begin{aligned} e^{Ax} &= Pe^{Dx}P^{-1} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2x} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} e^{2x} & e^{-x} - e^{2x} & e^{-x} - e^{2x} \\ -1 + e^{2x} & 2 - e^{2x} & 1 - e^{2x} \\ 1 - e^{2x} & e^{-x} - 2 + e^{2x} & e^{-x} - 1 + e^{2x} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} Y = e^{Ax}\Gamma = e^{Ax} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \\ = \begin{bmatrix} (\gamma_2 + \gamma_3)e^{-x} + (\gamma_1 - \gamma_2 - \gamma_3)e^{2x} \\ (-\gamma_1 + 2\gamma_2 + \gamma_3) + (\gamma_1 - \gamma_2 - \gamma_3)e^{2x} \\ (\gamma_2 + \gamma_3)e^{-x} + (\gamma_1 - 2\gamma_2 - \gamma_3) + (-\gamma_1 + \gamma_2 + \gamma_3)e^{2x} \end{bmatrix}. \end{aligned}$$

Now suppose we want to solve the initial value problem $Y' = AY$, $Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. *Then*

$$\begin{aligned} Y &= e^{Ax}\, Y(0) \\ &= \begin{bmatrix} e^{2x} & e^{-x} - e^{2x} & e^{-x} - e^{2x} \\ -1 + e^{2x} & 2 - e^{2x} & 1 - e^{2x} \\ 1 - e^{2x} & e^{-x} - 2 + e^{2x} & e^{-x} - 1 + e^{2x} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{2x} \\ -1 + e^{2x} \\ 1 - e^{2x} \end{bmatrix}. \end{aligned}$$

Remark C.8. Let us compare the results of our method here with that of our previous method. In the case of example 6, our previous method gives the solution

$$\begin{aligned} Y &= P \begin{bmatrix} e^{4x} & 0 \\ 0 & e^{-x} \end{bmatrix} C \\ &= Pe^{Dx}C \end{aligned}$$

where $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$,
while our method here gives

$$Y = Pe^{Dx}P^{-1}\Gamma.$$

But note that these answers are really the same! For P^{-1} is a constant matrix, so if Γ is a vector of arbitrary constants, then so is $P^{-1}\Gamma$, and we simply set $C = P^{-1}\Gamma$.

Similarly, in the case of example 7, our previous method gives the solution

$$\begin{aligned} Y &= P \begin{bmatrix} e^{-x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2x} \end{bmatrix} C \\ &= Pe^{Dx}C \end{aligned}$$

where $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$,
while our method here gives

$$Y = Pe^{Dx}P^{-1}\Gamma.$$

and again, setting $C = P^{-1}\Gamma$, we see that these answers are the same.

So the point here is not that the matrix exponential enables us to solve new problems, but rather that it gives a new viewpoint about the solutions we have already obtained.

While these two methods are in principle the same, we may ask which is preferable in practice. In this regard we see that our earlier method is better, as the use of the matrix exponential requires us to find P^{-1}, which may be a considerable amount of work. However, this advantage is (partially) negated if we wish to solve initial value problems, as the matrix exponential method immediately gives the unknown constants Γ, as $\Gamma = Y(0)$, while in the former method we must solve a linear system to obtain the unknown constants C.

Now let us consider the nondiagonalizable case. Suppose $Y' = JY$ where J is a matrix consisting of a single Jordan block. Then by theorem 2 this has the solution $Y = e^{Jx}\Gamma$. On the other hand, we already saw what the solution to this system was, in theorem 1 of section B. If we call the matrix there M, the solution is $Y = MC$. In this case we simply have $C = \Gamma$, so we must have $e^{Jx} = M$. Let us see that this is true by computing e^{Jx} directly.

Theorem C.9. *Let J be a k-by-k Jordan block with eigenvalue a,*

$$J = \begin{bmatrix} a & 1 & & & & \\ & a & 1 & & 0 & \\ & & a & 1 & & \\ & & & \ddots & \ddots & \\ & 0 & & & a & 1 \\ & & & & & a \end{bmatrix}.$$

Then

$$e^{Jx} = e^{ax} \begin{bmatrix} 1 & x & x^2/2! & x^3/3! & \cdots & x^{k-1}/(k-1)! \\ 0 & 1 & x & x^2/2! & \cdots & x^{k-2}/(k-2)! \\ & & 1 & x & \cdots & x^{k-3}/(k-3)! \\ & & & \ddots & & \vdots \\ & & & & & x \\ & & & & & 1 \end{bmatrix}.$$

Proof. First suppose that J is a 2-by-2 Jordan block,

$$J = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}.$$

Then $J^2 = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix}$, $J^3 = \begin{bmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{bmatrix}$, $J^4 = \begin{bmatrix} a^4 & 4a^3 \\ 0 & a^4 \end{bmatrix}, \ldots$
so

$$\begin{aligned} e^{Jx} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} x + \frac{1}{2!} \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix} x^2 + \frac{1}{3!} \begin{bmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{bmatrix} x^3 + \frac{1}{4!} \begin{bmatrix} a^4 & 4a^3 \\ 0 & a^4 \end{bmatrix} x^4 + \ldots \\ &= \begin{bmatrix} 1 + ax + \frac{1}{2!}(ax)^2 + \frac{1}{3!}(ax)^3 + \frac{1}{4!}(ax)^4 + \ldots & x + ax^2 + \frac{1}{2!}a^2x^3 + \frac{1}{3!}a^3x^4 + \ldots \\ 0 & 1 + ax + \frac{1}{2!}(ax)^2 + \frac{1}{3!}(ax)^3 + \frac{1}{4!}(ax)^4 + \ldots \end{bmatrix} \end{aligned}$$

We recognize the two diagonal entries as the power series for e^{ax}. As for the entry in the upper right-hand corner,

$$\begin{aligned} x + ax^2 + \frac{1}{2!}a^2x^3 + \frac{1}{3!}a^3x^4 + \ldots &= x(1 + ax + \frac{1}{2!}(ax)^2 + \frac{1}{3!}(ax)^3 + \ldots) \\ &= xe^{ax} \end{aligned}$$

and so we conclude that

$$e^{Jx} = \begin{bmatrix} e^{ax} & xe^{ax} \\ 0 & e^{ax} \end{bmatrix} = e^{ax}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

Next suppose that J is a 3-by-3 Jordan block,

$$J = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $J^2 = \begin{bmatrix} a^2 & 2a & 1 \\ 0 & a^2 & 2a \\ 0 & 0 & a^2 \end{bmatrix}$, $J^3 = \begin{bmatrix} a^3 & 3a^2 & 3a \\ 0 & a^3 & 3a^2 \\ 0 & 0 & a^3 \end{bmatrix}$, $J^4 = \begin{bmatrix} a^4 & 4a^3 & 6a^2 \\ 0 & a^4 & 4a^3 \\ 0 & 0 & a^4 \end{bmatrix}$, $J^5 = \begin{bmatrix} a^5 & 5a^4 & 10a^3 \\ 0 & a^5 & 5a^4 \\ 0 & 0 & a^5 \end{bmatrix}$, ...

so

$$\begin{aligned} e^{Jx} = &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} x + \frac{1}{2!}\begin{bmatrix} a^2 & 2a & 1 \\ 0 & a^2 & 2a \\ 0 & 0 & a^2 \end{bmatrix} x^2 + \frac{1}{3!}\begin{bmatrix} a^3 & 3a^2 & 3a \\ 0 & a^3 & 3a^2 \\ 0 & 0 & a^3 \end{bmatrix} x^3 \\ &+ \frac{1}{4!}\begin{bmatrix} a^4 & 4a^3 & 6a^2 \\ 0 & a^4 & 4a^3 \\ 0 & 0 & a^4 \end{bmatrix} x^4 + \frac{1}{5!}\begin{bmatrix} a^5 & 5a^4 & 10a^3 \\ 0 & a^5 & 5a^4 \\ 0 & 0 & a^5 \end{bmatrix} x^5 + \ldots \\ &= \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{bmatrix}, \end{aligned}$$

and we see that

$$\begin{aligned} m_{11} = m_{22} = m_{33} &= 1 + ax + \frac{1}{2!}(ax)^2 + \frac{1}{3!}(ax)^3 + \frac{1}{4!}(ax)^4 + \frac{1}{5!}(ax)^5 + \ldots \\ &= e^{ax}, \end{aligned}$$

and

$$\begin{aligned} m_{12} = m_{23} &= x + ax^2 + \frac{1}{2!}a^2x^3 + \frac{1}{3!}a^3x^4 + \frac{1}{4!}a^4x^5 + \ldots \\ &= xe^{ax} \end{aligned}$$

as we saw in the 2-by-2 case. Finally,

$$\begin{aligned} m_{13} &= \frac{1}{2!}x^2 + \frac{1}{2!}ax^3 + \frac{1}{2!}(\frac{1}{2!}a^2x^4) + \frac{1}{2!}(\frac{1}{3!}a^3x^5) + \dots \\ &\text{(as } 6/4! = 1/4 = (1/2!)(1/2!) \text{ and } 10/5! = 1/12 = (1/2!)(1/3!)) \\ &= \frac{1}{2!}x^2(1 + ax + \frac{1}{2!}(ax)^2 + \frac{1}{3!}(ax)^3 + \dots) \\ &= \frac{1}{2!}x^2e^{ax}, \end{aligned}$$

so

$$e^{Jx} = \begin{bmatrix} e^{ax} & xe^{ax} & \frac{1}{2!}x^2e^{ax} \\ 0 & e^{ax} & xe^{ax} \\ 0 & 0 & e^{ax} \end{bmatrix} = e^{ax} \begin{bmatrix} 1 & x & x^2/2! \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix},$$

and similarly for larger Jordan blocks. □

Let us see how to apply this theorem in a couple of examples.

Example C.10. *(Compare examples 3 and 7 section B). Consider the system*

$$Y' = AY \qquad \textit{where} \qquad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

Also, consider the initial value problem $Y' = AY$, $Y(0) = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$.

We saw in section A example 9 that $A = PJP^{-1}$ *with*

$$P = \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix} \quad \textit{and} \quad J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then

$$\begin{aligned} e^{Ax} &= Pe^{Jx}P^{-1} \\ &= \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} e^{2x} & xe^{2x} \\ 0 & e^{2x} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (1-2x)e^{2x} & xe^{2x} \\ -4xe^{2x} & (1+2x)e^{2x} \end{bmatrix}, \end{aligned}$$

and so

$$\begin{aligned} Y &= e^{Ax}\Gamma \\ &= e^{Ax}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} \\ &= \begin{bmatrix}\gamma_1 e^{2x} + (-2\gamma_1+\gamma_2)xe^{2x} \\ \gamma_2 e^{2x} + (-4\gamma_1 + 2\gamma_2)xe^{2x}\end{bmatrix}. \end{aligned}$$

The initial value problem has solution

$$\begin{aligned} Y &= e^{Ax}Y_0 \\ &= e^{Ax}\begin{bmatrix}3\\ -8\end{bmatrix} \\ &= \begin{bmatrix}3e^{2x} - 14xe^{2x} \\ 8e^{2x} - 28xe^{2x}\end{bmatrix}. \end{aligned}$$

Example C.11. *(Compare examples 4 and 8 of section B). Consider the system*

$$Y' = AY \qquad \text{where} \qquad A = \begin{bmatrix}2 & 1 & 1\\ 2 & 1 & -2\\ -1 & 0 & 2\end{bmatrix}.$$

Also, consider the initial value problem $Y' = AY$, $Y(0) = \begin{bmatrix}8\\ 32\\ 5\end{bmatrix}$.

We saw in section A example 17 that $A = PJP^{-1}$ *with*

$$P = \begin{bmatrix}1 & 0 & -5\\ -2 & 0 & 6\\ -1 & 1 & 1\end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 3\end{bmatrix}.$$

Then

$$\begin{aligned} e^{Ax} &= Pe^{Jx}P^{-1} \\ &= \begin{bmatrix}1 & 0 & -5\\ -2 & 0 & 6\\ -1 & 1 & 1\end{bmatrix}\begin{bmatrix}e^{-x} & xe^{-x} & 0\\ 0 & e^{-x} & 0\\ 0 & 0 & e^{3x}\end{bmatrix}\begin{bmatrix}1 & 0 & -5\\ -2 & 0 & 6\\ -1 & 1 & 1\end{bmatrix}^{-1} \\ &= \begin{bmatrix}\frac{3}{8}e^{-x} + \frac{1}{2}xe^{-x} + \frac{5}{8}e^{3x} & -\frac{5}{16}e^{-x} - \frac{1}{4}xe^{-x} + \frac{5}{16}e^{3x} & xe^{-x} \\ -\frac{3}{4}e^{-x} - xe^{-x} + \frac{3}{4}e^{3x} & \frac{5}{8}e^{-x} + \frac{1}{2}xe^{-x} + \frac{3}{8}e^{3x} & -2xe^{-x} \\ \frac{1}{8}e^{-x} - \frac{1}{2}xe^{-x} - \frac{1}{8}e^{3x} & \frac{1}{16}e^{-x} + \frac{1}{4}xe^{-x} - \frac{1}{16}e^{3x} & e^{-x} - xe^{-x}\end{bmatrix} \end{aligned}$$

and so

$$\begin{aligned} Y &= e^{Ax}\Gamma \\ &= e^{Ax}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix} \\ &= \begin{bmatrix} (\frac{3}{8}\gamma_1 - \frac{5}{16}\gamma_2)e^{-x} + (\frac{1}{2}\gamma_1 - \frac{1}{4}\gamma_2 + \gamma_3)xe^{-x} + (\frac{5}{8}\gamma_1 + \frac{5}{16}\gamma_2)e^{3x} \\ (-\frac{3}{4}\gamma_1 + \frac{5}{8}\gamma_2)e^{-x} + (-\gamma_1 + \frac{1}{2}\gamma_2 - 2\gamma_3)xe^{-x} + (\frac{3}{4}\gamma_1 + \frac{3}{8}\gamma_2)e^{3x} \\ (\frac{1}{8}\gamma_1 + \frac{1}{16}\gamma_2 + \gamma_3)e^{-x} + (-\frac{1}{2}\gamma_1 + \frac{1}{4}\gamma_2 - \gamma_3)xe^{-x} + (-\frac{1}{8}\gamma_1 - \frac{1}{16}\gamma_2)e^{3x} \end{bmatrix}. \end{aligned}$$

The initial value problem has solution

$$\begin{aligned} Y &= e^{Ax}Y_0 \\ &= e^{Ax}\begin{bmatrix}8\\ 32\\ 5\end{bmatrix} \\ &= \begin{bmatrix} -7e^{-x} + xe^{-x} + 15e^{3x} \\ 14e^{-x} - 2xe^{-x} + 18e^{3x} \\ 8e^{-x} - xe^{-x} - 3e^{3x} \end{bmatrix}. \end{aligned}$$

Remark C.12. Our procedure in this section is essentially that of remark 9 of section B. (Compare example 11 with example 10 of section B.)

C. Exercises

1-20. For each exercise, see the corresponding exercise in section B.

(a) Find e^{Ax} and the solution $Y = e^{Ax}\Gamma$ of $Y' = AY$.

(b) Use part (a) to solve the initial value problem $Y' = AY, Y(0) = Y_0$. (Of course, the answer to part (b) is the same as that in section B.)

C. Answers to exercises

1a. $$e^{Ax} = \begin{bmatrix} -35e^{3x} + 36e^{5x} & -28e^{3x} + 28e^{5x} \\ 45e^{3x} - 45e^{5x} & 36e^{3x} - 35e^{5x} \end{bmatrix}$$

$$Y = \begin{bmatrix} (-35\gamma_1 - 28\gamma_2)e^{3x} + (36\gamma_1 + 28\gamma_2)e^{5x} \\ (45\gamma_1 + 36\gamma_2)e^{3x} + (-45\gamma_1 - 35\gamma_2)e^{5x} \end{bmatrix}$$

3a. $$e^{Ax} = \begin{bmatrix} e^{3x} - 21xe^{3x} & 9xe^{3x} \\ -49xe^{3x} & e^{3x} + 21xe^{3x} \end{bmatrix}$$

$$Y = \begin{bmatrix} \gamma_1 e^{3x} + (-21\gamma_1 + 9\gamma_2)xe^{3x} \\ \gamma_2 e^{3x} + (-49\gamma_1 + 21\gamma_2)xe^{3x} \end{bmatrix}$$

5a. $$e^{Ax} = \begin{bmatrix} e^{7x} - 5xe^{7x} & xe^{7x} \\ -25xe^{7x} & e^{7x} + 5xe^{7x} \end{bmatrix}$$

$$Y = \begin{bmatrix} \gamma_1 e^{7x} + (-5\gamma_1 + \gamma_2)xe^{7x} \\ \gamma_2 e^{7x} + (-25\gamma_1 + 5\gamma_2)xe^{7x} \end{bmatrix}$$

7a. $$e^{Ax} = \begin{bmatrix} e^{x} & 0 & 0 \\ -\frac{1}{2}e^{-x} + \frac{1}{2}e^{x} & \frac{1}{4}e^{-x} + \frac{3}{4}e^{3x} & \frac{3}{4}e^{-x} - \frac{3}{4}e^{3x} \\ -\frac{1}{2}e^{-x} + \frac{1}{2}e^{x} & \frac{1}{4}e^{-x} - \frac{1}{4}e^{3x} & \frac{3}{4}e^{-x} + \frac{1}{4}e^{3x} \end{bmatrix}$$

$$Y = \begin{bmatrix} \gamma_1 e^{x} \\ (-\frac{1}{2}\gamma_1 + \frac{1}{4}\gamma_2 + \frac{3}{4}\gamma_3)e^{-x} + \frac{1}{2}\gamma_1 e^{x} + (\frac{3}{4}\gamma_2 - \frac{3}{4}\gamma_3)e^{3x} \\ (-\frac{1}{2}\gamma_1 + \frac{1}{4}\gamma_2 + \frac{3}{4}\gamma_3)e^{-x} + \frac{1}{2}\gamma_1 e^{x} + (-\frac{1}{4}\gamma_2 + \frac{1}{4}\gamma_3)e^{3x} \end{bmatrix}$$

9a. $$e^{Ax} = \begin{bmatrix} -e^{-3x} + 2e^{x} & -2e^{-3x} + 2e^{x} & -4e^{-3x} + 4e^{x} \\ -e^{-3x} + e^{x} & e^{x} & -2e^{-3x} + 2e^{x} \\ e^{-3x} - e^{x} & e^{-3x} - e^{x} & 3e^{-3x} - 2e^{x} \end{bmatrix}$$

$$Y = \begin{bmatrix} (-\gamma_1 - 2\gamma_2 - 4\gamma_3)e^{-3x} + (2\gamma_1 + 2\gamma_2 + 4\gamma_3)e^{x} \\ (-\gamma_1 - 2\gamma_3)e^{-3x} + (\gamma_1 + \gamma_2 + 2\gamma_3)e^{x} \\ (\gamma_1 + \gamma_2 + 2\gamma_3)e^{-3x} + (-\gamma_1 - \gamma_2 - 2\gamma_3)e^{x} \end{bmatrix}$$

11a. $$e^{Ax} = \begin{bmatrix} \frac{3}{2}e^{4x} - \frac{1}{2}e^{2x} & e^{4x} - e^{2x} & \frac{1}{2}e^{4x} - \frac{1}{2}e^{2x} \\ -\frac{1}{2}e^{4x} + \frac{1}{2}e^{2x} & e^{2x} & -\frac{1}{2}e^{4x} + \frac{1}{2}e^{2x} \\ -\frac{1}{2}e^{4x} + \frac{1}{2}e^{2x} & -e^{4x} + e^{2x} & \frac{1}{2}e^{4x} + \frac{1}{2}e^{2x} \end{bmatrix}$$

$$Y = \begin{bmatrix} (\frac{3}{2}\gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3)e^{4x} + (-\frac{1}{2}\gamma_1 - \gamma_2 - \frac{1}{2}\gamma_3)e^{2x} \\ (-\frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_3)e^{4x} + (\frac{1}{2}\gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3)e^{2x} \\ (-\frac{1}{2}\gamma_1 - \gamma_2 + \frac{1}{2}\gamma_3)e^{4x} + (\frac{1}{2}\gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3)e^{2x} \end{bmatrix}$$

13a. $$e^{Ax} = \begin{bmatrix} e^{-2x} - xe^{-2x} & xe^{-2x} & -xe^{-2x} \\ e^{-2x} - xe^{-2x} - e^{4x} & xe^{-2x} + e^{4x} & -xe^{-2x} \\ e^{-2x} - e^{4x} & -e^{-2x} + e^{4x} & e^{-2x} \end{bmatrix}$$

$$Y = \begin{bmatrix} \gamma_1 e^{-2x} + (-\gamma_1 + \gamma_2 - \gamma_3)xe^{-2x} \\ \gamma_1 e^{-2x} + (-\gamma_1 + \gamma_2 - \gamma_3)xe^{-2x} + (-\gamma_1 + \gamma_2)e^{4x} \\ (\gamma_1 - \gamma_2 + \gamma_3)e^{-2x} + (-\gamma_1 + \gamma_2)e^{4x} \end{bmatrix}$$

15a. $$e^{Ax} = \begin{bmatrix} 3 - 2e^{3x} & 9x & 2 + 6x - 2e^{3x} \\ 2 - 2e^{3x} & 1 + 6x & 2 + 4x - 2e^{3x} \\ -3 + 3e^{3x} & -9x & -2 - 6x + 3e^{3x} \end{bmatrix}$$

$$Y = \begin{bmatrix} (3\gamma_1 + 2\gamma_3) + (9\gamma_2 + 6\gamma_3)x + (-2\gamma_1 - 2\gamma_3)e^{3x} \\ (2\gamma_1 + \gamma_2 + 2\gamma_3) + (6\gamma_2 + 4\gamma_3)x + (-2\gamma_1 - 2\gamma_3)e^{3x} \\ (-3\gamma_1 - 2\gamma_3) + (-9\gamma_2 - 6\gamma_3)x + (3\gamma_1 + 3\gamma_3)e^{3x} \end{bmatrix}$$

17a. $$e^{Ax} = \begin{bmatrix} e^x - 2xe^x & xe^x & -xe^x \\ -10xe^x & e^x + 5xe^x & -5xe^x \\ -6xe^x & 3xe^x & e^x - 3xe^x \end{bmatrix}$$

$$Y = \begin{bmatrix} \gamma_1 e^x + (-2\gamma_1 + \gamma_2 - \gamma_3)xe^x \\ \gamma_2 e^x + (-10\gamma_1 + 5\gamma_2 - 5\gamma_3)xe^x \\ \gamma_3 e^x + (-6\gamma_1 + 3\gamma_2 - 3\gamma_3)xe^x \end{bmatrix}$$

19a. $$e^{Ax} = \begin{bmatrix} 1 - 4x - \frac{3}{2}x^2 & x + \frac{1}{2}x^2 & 2x + \frac{1}{2}x^2 \\ -5x - 3x^2 & 1 + x + x^2 & 3x + x^2 \\ -7x - \frac{3}{2}x^2 & 2x + \frac{1}{2}x^2 & 1 + 3x + \frac{1}{2}x^2 \end{bmatrix}$$

$$Y = \begin{bmatrix} \gamma_1 + (-4\gamma_1 + \gamma_2 + 2\gamma_3)x + (-\frac{3}{2}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\gamma_3)x^2 \\ \gamma_2 + (-5\gamma_1 + \gamma_2 + 3\gamma_3)x + (-3\gamma_1 + \gamma_2 + \gamma_3)x^2 \\ \gamma_3 + (-7\gamma_1 + 2\gamma_2 + 3\gamma_3)x + (-\frac{3}{2}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\gamma_3)x^2 \end{bmatrix}$$