Math 205, Summer I 2009
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Week 6:

[reprint, week4: 5.4 Eigenvalues and Eigenvectors] + diagonaliization

5.5 Eigenspaces and Diagonalization

6.2 Homogeneous systems of first order linear DE

A vector $\vec{v} \neq \vec{0}$ in $\mathbf{R}^n$ (or in $\mathbf{C}^n$)

is an **eigenvector with eigenvalue** λ of an n-by-n matrix A if $A\vec{v} = \lambda\vec{v}$.

We re-write the vector equation as $(A - \lambda I_n)\vec{v} = \vec{0}$,

which is a homogeneous system with coef matrix $(A - \lambda I)$, and we want λ

so that the system has a non-trivial solution.

We see that the eigenvalues are the roots

of the **characteristic polynomial** $P(\lambda) = 0$, where $P(\lambda) = \det(A - \lambda I)$.

To find the eigenvectors we find the distinct

roots $\lambda = \lambda_i$, and for each i solve $(A - \lambda_i I)\vec{v} = \vec{0}$.

Problem 1. Find the eigenvalues and eigenvectors

of $A = \begin{pmatrix} 3 & -1 \\ -5 & -1 \end{pmatrix}$

Solution.

$$P(\lambda) = det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & -1 \\ -5 & -1-\lambda \end{vmatrix}$$

$= (-1 - \lambda)(3 - \lambda) - 5 = \lambda^2 - 2\lambda - 3 - 5$

$= \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$, so $\lambda_1 = 4, \lambda_2 = -2$.

For $\lambda_1 = 4$, we reduce the coef matrix of the system $(A - 4I)\vec{x} = \vec{0}$,

$$A - 4I = \begin{pmatrix} 3-4 & -1 \\ -5 & -1-4 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -5 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so for $x_2 = a$ we get $x_1 = -a$, and the eigenvectors for $\lambda_1 = 4$ are the vectors $(x_1, x_2) = a(-1, 1)$ with $a \neq 0$, from the space with basis $\vec{v}_1 = (-1, 1)$.

For $\lambda_2 = -2$, we start over with coef

$$A + 2I = \begin{pmatrix} 3+2 & -1 \\ -5 & -1+2 \end{pmatrix} \to \begin{pmatrix} 5 & -1 \\ 0 & 0 \end{pmatrix}.$$

To find a spanning set without fractions, we anticipate that finding x_1 will use division by 5, and take $x_2 = 5b$. Then $5x_1 - x_2 = 0$ gives $5x_1 = 5b$, so $x_1 = b$, and $(x_1, x_2) = b(1, 5)$, spanned by $\vec{v}_2 = (1, 5)$ [or, if you must, $(\frac{1}{5}, 1)$].

Example 1 (diagonalization): We write $\vec{v}_1, \vec{v}_2$ as columns, then

$$P^{-1}AP = D \text{ with } P = \begin{pmatrix} -1 & 1 \\ 1 & 5 \end{pmatrix} \text{ and } D = diag(4, -2) = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Problem 2.

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 10 & -12 & 8 \\ 0 & 2 & 0 \\ -8 & 12 & -6 \end{pmatrix}$.

Solution.

Since A is 3-by-3, the characteristic polynomial $P(\lambda)$ is cubic, ...

... expansion on the 2nd row gives:

we get $P(\lambda) = (2 - \lambda) \begin{vmatrix} 10-\lambda & 8 \\ -8 & -6-\lambda \end{vmatrix}$

$$= (2-\lambda)[(10-\lambda)(-6-\lambda) + 64] = (2-\lambda)[\lambda^2 - 4\lambda - 60 + 64] = -(\lambda - 2)^3.$$

Anyway, this matrix only has one distinct

eigenvalue, $\lambda_1 = 2$. To find the eigenvectors,

we reduce the coef matrix $A - 2I$

$$= \begin{pmatrix} 10-2 & -12 & 8 \\ 0 & 2-2 & 0 \\ -8 & 12 & -6-2 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that x_2 and x_3 will be free variables,

and anticipating that finding x_1 will involve

division by 2, take $x_2 = 2s$ and $x_3 = t$, so

$x_1 = 3s - t$; and $(x_1, x_2, x_3) = (3s - t, 2s, t)$

$= s(3, 2, 0) + t(-1, 0, 1)$, has LI spanning vectors

$(3, 2, 0)$ and $(-1, 0, 1)$, giving eigenvectors except when $s = t = 0$.

Example 2 (diagonalization): This 3-by-3 matrix is

NOT diagonalizable, since either

(1) we need 3 LI eigenvectors, but only have 2; OR

(2) the repeated eigenvalue $\lambda_1 = 2$ has multiplicity $m_1 = 3$, but the

dimension of the eigenspace E_2 is $d_1 = 2$, with $m_1 \neq d_1$.

Problem 3. Find the eigenvalues and eigenvectors

of $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$.

Solution.

$$P(\lambda) = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = [(3-\lambda)(-1-\lambda) + 8] = \lambda^2 - 2\lambda - 3 + 8$$

$= \lambda^2 - 2\lambda + 5$, with roots $\lambda = 1 \pm 2i$. So there are no real λ for which there is a non-trivial real solution $\vec{x} \in \mathbf{R}^2$. But there is a complex solution $\vec{0} \neq \vec{x} \in \mathbf{C}^2$, and we solve for $\vec{x}$ by reducing the coef matrix.

The eigenvalues are complex conjugates, say
$\lambda_1 = 1 - 2i, \lambda_2 = \overline{\lambda_1} = 1 + 2i$.

We only need to solve one of the systems.
For $\lambda_1 = 1 - 2i$, $A - (1 - 2i)I =$

$$\begin{pmatrix} 2+2i & -2 \\ 4 & -2+2i \end{pmatrix} \to \begin{pmatrix} 2 & -1+i \\ 0 & 0 \end{pmatrix}.$$

We can do this calculation direcly.
As basis for the solution we take $\vec{v}_1 = (1 - i, 2)$,
and then get $\vec{v}_2 = (1 + i, 2)$ for λ_2,
since an eigenvector for the conjugate value is the
conjugate vector (when A has real entries).

Example 3 (diagonalization): The eigenvalues are distinct,
so this matrix is diagonalizable, and the matrix P uses (complex) eigenvectors
$\vec{v}_1, \vec{v}_2$ just as in Example 1.

We will see that an n-by-n matrix A is **diagonalizable**
if and only if A has n linearly independent eigenvectors.
First, we have a new definition, matrices
A and B are **similar** if there is an n-by-n
matrix P so that P has an inverse and $B = P^{-1}AP$.

We also recall that a **diagonal matrix** $D = diag(d_1, \ldots, d_n)$

is a square matrix with all entries 0 except (possibly) on the main diagonal, where the entries are $d_1, \ldots, d_n$ (in that order).

Finally, a matrix A is **diagonalizable** if there is a matrix P so that $P^{-1}AP = D$ is a diagonal matrix.

Theorem. The matrix A is diagonalizable exactly when

A has n linearly independent eigenvectors.

Further, (1) the diagonal entries in the diagonalization are the eigenvalues, each occurring as many times on the diagonal as the multiplicity m_j;

(2) the columns of the matrix P that diagonalizes A are n LI eigenvectors of A; and,

(3) the j-th column of P has eigenvalue that is the j-th entry on the diagonal.

Problem 4.

For each eigenvalue, find the multiplicity and a basis for the eigenspace and determine whether

$$A = \begin{pmatrix} 4 & 1 & 6 \\ -4 & 0 & -7 \\ 0 & 0 & -3 \end{pmatrix} \text{ is diagonalizable.}$$

Solution.

Expanding $\det(A - \lambda I)$ on the 3rd row, we have $P(\lambda) = (-3 - \lambda) \begin{vmatrix} 4-\lambda & 1 \\ -4 & -\lambda \end{vmatrix}$,

$= -(3+\lambda)[-4\lambda + \lambda^2 + 4] = -(3+\lambda)(\lambda - 2)^2$,

so $\lambda_1 = 2$ is an eigenvalue with multiplicity 2,

and $\lambda_2 = -3$ is a simple root (multiplicity 1).

For $\lambda_1 = 2$, $A - 2I = \begin{pmatrix} 2 & 1 & 6 \\ -4 & -2 & -7 \\ 0 & 0 & -5 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

We already have sufficient info to determine

that A is not diagonalizable: Rank($A - 2I$)=2, so

the dimension of the eigenspace (nullspace!) is 3-2 =1,

less than the multiplicity. We still need bases of the eigenspaces;

and see that $\vec{v}_1 = (-1, 2, 0)$ is a basis for $\lambda_1 = 2$.

For $\lambda_2 = -3$, $A - (-3I) = A + 3I = \begin{pmatrix} 7 & 1 & 6 \\ -4 & 3 & -7 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 7 & -8 \\ 0 & -25 & 25 \\ 0 & 0 & 0 \end{pmatrix}$

(we use $R_1 \to R_1 + 2R_2$; update, then use $R_2 \to R_2 - 4R_1$)

$\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ ($R_1 \to -R_1, R_2 \to -\frac{1}{25}R_2$; update, then $R_1 \to R_1 + 7R_2$.)

So a basis is $\vec{v}_2 = (-1, 1, 1)$, dim(E_{-3})=3-2=1.

Problem 5.

Determine whether $A = \begin{pmatrix} 1 & -3 & 1 \\ -1 & -1 & 1 \\ -1 & -3 & 3 \end{pmatrix}$ is diagonalizable or not,

given $P(\lambda) = (\lambda - 2)^2(\lambda + 1)$.

Solution.

Since $\lambda_2 = -1$ is a simple root it is non-defective;

and we only need to check $\lambda_1 = 2$. We have $A - 2I = \begin{pmatrix} -1 & -3 & 1 \\ -1 & -3 & 1 \\ -1 & -3 & 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $\dim(E_2) = 3 - 1 = 2$, and A is diagonalizable.

We're not asked for a basis here, but for practice,

$x_2 = a, x_3 = b$ and $x_1 + 3x_2 - x_3 = 0$ gives $x_1 = -3a + b$,

$(x_1, x_2, x_3) = (-3a + b, a, b) = (-3a, a, 0) + (b, 0, b) = a(-3, 1, 0) + b(1, 0, 1)$,

so a basis is $\vec{v}_1 = (-3, 1, 0)$ and $\vec{v}_2 = (1, 0, 1)$.

Example 5 (diagonalization): We write $\vec{v}_1, \vec{v}_2$ as columns,

and also $v_3 = (1, 1, 1)$ as a column for $\lambda = -1$, then

$P = (v_1 v_2 v_3)$ gives $A = PDP^{-1}$, with $D = diag(2, 2, -1)$.

Finally, we collect the terminology and results used. If $P(\lambda)$ written in factored form is $P(\lambda) = (\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_r)^{m_r}$, where the λ_j are the distinct roots, we say that λ_j has (algebraic) multiplicity m_j, or that λ_j is a simple root (multiplicity 1) if $m_j = 1$.

Let $d_j = \dim(E_{\lambda_j})$ be the dimension of the eigenspace (sometimes called the geometric multiplicity), then the main facts are

1. $1 \leq d_j \leq m_j$;
2. A is non-diagonalizable exactly when there is some eigenvalue (which must be a repeated root) that is defective, $d_j < m_j$;
3. A is diagonalizable exactly when every eigenvalue is non-defective, $d_j = m_j$, for $j = 1, \ldots, r$;
4. In particular, if $P(\lambda)$ has distinct roots then A is always diagonalizable $(1 \leq d_j \leq m_j = 1)$.

We recall that m_j - the algebraic multiplicity - is the number of times the jth distinct eigenvalue is a root of the characteristic polynomial; and d_j - the geometric multiplicity - is the dimension of the λ_jth eigenspace (that is, the nullspace of $A - \lambda_j I$).

Problem 6. Compare eigenspaces and the

defective condition for $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{pmatrix}$ and

$A_1 = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{pmatrix}.$

Solution. Both matrices have char polyn $P(\lambda) = -(\lambda - 4)^2(\lambda + 1)$.

Since $\lambda = -1$ a simple root (not repeated), it is non-defective.

We have $A - 4I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$A_1 - 4I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$

These matrices test our eigenvector-finding-skills. For the 2nd, the equations say $x_2 = x_3 = 0$; while we must have a nontrivial solution. Do you see one?

Don't Think! Our reflex is x_2, x_3 bound; $x_1 = a$ free,

so $(x_1, x_2, x_3) = (a, 0, 0) = a(1, 0, 0)$.

We also have $\vec{v}_1 = (1, 0, 0)$ an eigenvector for the first matrix, but there's also a 2nd LI solution, $\vec{v}_2 = (0, -3, 2)$. So the roots and multiplicities are the same, only one entry is changed, but A is diagonalizable; while A_1 is non-diagonalizable.

6.2 Systems of first order linear DE

We apply this to solve systems of first

order DE with diaonalizable coef matrix.

Problem 7.

Solve $x_1' = \quad 9x_1 + 6x_2$

$x_2' = -10x_1 - 7x_2$

using the method of 6.2 (compare: 6.1 Hw).

Solution.

We first need the eigenvectors to find the matrix P. We have

$P(\lambda) = (9 - \lambda)(-7 - \lambda) + 60 = \lambda^2 - 2\lambda - 63 + 60$

$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$,

so $\lambda_1 = -1, \lambda_2 = 3$. For $\lambda_1 = -1$,

$$A + I = \begin{pmatrix} 10 & 6 \\ -10 & -6 \end{pmatrix} \to \begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix},$$

and $\vec{v}_1 = (-3, 5)$. For $\lambda_2 = 3$,

$$A - 3I = \begin{pmatrix} 6 & 6 \\ -10 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

and $\vec{v}_2 = (-1, 1)$. Writing the eigenvectors

as columns, $P = \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix}$ will have $P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

Now the "method of Section 6.2" is to

make the change-of-variables $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

giving the new system of DE $\vec{y}' = P^{-1}AP\vec{y}$, which is the "uncoupled system"

$y_1' = -y_1$

$y_2' = 3y_2.$

The solution is $y_1 = c_1 e^{-t}, y_2 = c_2 e^{3t}$ so

the original system had solution $\vec{x} = P\vec{y}$, which we write as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{3t} \end{pmatrix}$$

$$= c_1 e^{-t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Notice that this solution is a linear combination

of two LI solutions of the form $e^{\lambda_j t} \vec{v}_j$,

where $\vec{v}_j$ is an eigenvector with eigenvalue

λ_j, as discussed in class, as motivation for eigenvectors.