Week 3b:

3.6 Cooling/Mixing [see Cooling example, week 3]

[3.9 Numerical Solutions [Euler, Maple's dsolve/numeric]] not collected

4.1 Higher Order DE

4.2 Constant Coef, Homogeneous DE

Problem: A 200L tank is half full of a solution containing 100g of a desolved chemical. A solution containing 0.5 g/L of the same chemical is pumped into the tank at a rate of 6 L/min. The well-stirred mixture is pumped out at a rate of 4 L/min. Determine the concentration of the chemical in the tank just before overflow.

Solution:

$V = V(t)$ and $A = A(t)$ will be the volumn of the solution in the tank (in L) and amount of chemical in the tank (in g), at time t (in min.). The "rate in" is, $r_1 = 6$ and "rate out", $r_2 = 4$; and the "concentration in" is, $c_1 = 0.5$, while we solve for the "concentration out", $c_2 = \frac{A}{V}$.

First, the initial volumn $V_0 = 100$ (one half of the tank's volumn), so $V(t) = 100 + (r_1 - r_2)t = 100 + 2t$ ("rate in" - "rate out"). Next, the tank overflows when V is 200, which occurs when $t = 50$, (why?)

so we're looking for $c_2(50)$.

Now our main DE says that the rate of change of the amount of the chemical, $\frac{dA}{dt}$, is the difference $r_1c_1 - r_2c_2$ ("rate in" - "rate out"), which we re-write as

$$\frac{dA}{dt} + \left(\frac{4}{100+2t}\right) A = 3, \text{ (using } c_2 = \tfrac{A}{V}).$$

Now $\frac{dA}{dt} + \left(\frac{4}{100+2t}\right) A = 3$ is a Linear DE,

with coefficient $P = \frac{4}{100+2t} = \frac{2}{50+t}$.

So $\int P\,dt = 2\ln(50+t) = \ln(50+t)^2$,

and the integral factor $I = e^{\int P\,dt} = e^{\ln(50+t)^2}$

$= (50+t)^2$. Multiplying both sides by I and using the Main Property gives

$$\tfrac{d}{dt}\left((50+t)^2 A\right) = 3(50+t)^2,$$

then integrating gives $(50+t)^2 A = (50+t)^3 + c$, so

$$A = 50 + t + \frac{c}{(50+t)^2}.$$

One last bit of data, the initial amount of the chemical was 100g, which gives the initial condition $A(0) = 100$,

so $100 = 50 + \frac{c}{50^2}$, and $c = 50^3$,

$$A(t) = 50 + t + \frac{50^3}{(50+t)^2}.$$

So from $A(t) = 50 + t + \dfrac{50^3}{(50+t)^2}$, we have $c_2(50) = \dfrac{A(50)}{V(50)}$

$= \dfrac{100 + \frac{125000}{100^2}}{200} = \dfrac{100 + \frac{125}{10}}{200} = \dfrac{100 + \frac{25}{2}}{200} = \dfrac{\frac{225}{2}}{200} = \dfrac{225}{400} = \dfrac{9}{16}$ g/L for the concentration at the time of overflow.

Logistic model: $P = P(t)$ pop. at time t,

$\dfrac{dP}{dt} = r(1 - \dfrac{P}{C})P$ where C is the carrying capacity (asymptotic limit (max) population), and r is the birth rate per individual; t in years.

Key Step in method: After $C\,dP = r(C-P)P\,dt$,

separation gives $\dfrac{C\,dP}{(C-P)P} = r\,dt$,

where we use Partial Fractions to get

$\dfrac{C}{(C-P)P} = \dfrac{A}{P} + \dfrac{B}{C-P}$ with $A = B = 1$,

so $\displaystyle\int \left(\frac{1}{P} + \frac{1}{C-P}\right) dP = rt + c_1$, where the integral is

$\ln P - \ln(C-P) = \ln \dfrac{P}{C-P}$.

Not-so Key Steps: Raising to e gives $\dfrac{P}{C-P} = e^{rt}e^{c_1}$,

where we replace e^{c_1} by c_2, and solve for P, then

use the initial condition to replace c_2 by $\dfrac{P_0}{C-P_0}$

and clear fractions to get the solution in a standard form.

Problem Setup: In practice, we don't use formulas, and simply observe that if the maximum population, the carrying capacity, is everyone, say $C = 1500$, and the DE is

$\frac{dP}{dt} = kP(1500 - P)$, we solve using Partial Fractions, without needing any other formulas.

General Solution: (1) to solve the 2nd order linear DE $y'' + a_1 y' + a_2 y = 0$

find two linearly independent solutions y_1 and y_2, then the general solution is $y_H = c_1 y_1 + c_2 y_2$.

(2) to solve the non-homogeneous 2nd order linear DE $y'' + a_1 y' + a_2 y = F$,

find a particular solution y_p, then the general solution is $y = y_H + y_p$, where y_H is the solution of (1).

Problem 1:

Determine all values r so $y = e^{rx}$ is a solution to $y'' - 4y' + 3y = 0$.

Find the general solution.

Solution:

For $y = e^{rx}$, we have $y' = re^{rx}$ and $y'' = r^2 e^{rx}$.

Substitution in the DE gives

$0 = y'' - 4y' + 3y = r^2 e^{rx} - 4re^{rx} + 3e^{rx}$

$= (r^2 - 4r + 3)e^{rx}$.

Now $r^2 - 4r + 3 = (r-3)(r-1) = 0$ has roots $r_1 = 1$ and $r_2 = 3$, so we get solutions $y_1 = e^x$ and $y_2 = e^{3x}$, so the general solution is $y = c_1 e^x + c_2 e^{3x}$.

Constant Coef, Homogeneous DE

When the DE $y'' + a_1 y' + a_2 y = 0$ has coefficients a_1 and a_2 that are constant, the two linearly independent solutions y_1 and y_2 in the general solution $y_H = c_1 y_1 + c_2 y_2$ may be determined using the roots of the characteristic polynomial $P(r) = r^2 + a_1 r + a_2$.

Three cases:

(1) For distinct real roots r_1 and r_2,
$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

(2) For a repeated real root r_1 the 2nd independent solution is $y_2 = x e^{r_1 x}$, and $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$.

(3) For a pair of complex roots $r = a + bi$ and $r = a - bi$ with a and b real, $b > 0$,
$y = c_1 e^{ax} \cos(bx) + c_2 e^{ax} \sin(bx)$.

Recitation on 4.2.

Problem 2:

Solve $y'' - 6y' + 34y = 0$.

Solution:

The characteristic polynomial is $r^2 - 6r + 34$, so the quadratic formula gives roots $\dfrac{6 \pm \sqrt{6^2 - 4 \cdot 34}}{2}$

$= 3 \pm 5i$. (why?)

So $y = e^{3x}(c_1 \cos 5x + c_2 \sin 5x)$ is the general solution.

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***Postscript:** We may

solve the IVP $y'' + 4y' + 4y = 0,\ y(0) = 1, y'(0) = 4$.

We also may solve an IVP for Problem 2 (time

permitting). In any case, the calculation of constants

c_1 and c_2 to solve an IVP look rather different in

(at least) four cases: the three types of roots in the

homogeneous DE, and

(fourth) the non-homogeneous DE.

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