

1. **Solve** $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 2$
 $2x_1 + 4x_2 + 9x_3 + 14x_4 + 22x_5 = 7$
 $5x_1 + 10x_2 + 17x_3 + 25x_4 + 36 = 12$

Solution: $(A|\vec{b}) \rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 2 \\ 0 & 0 & 3 & 6 & 12 & 3 \\ 0 & 0 & 2 & 5 & 11 & 2 \end{array} \right) \rightarrow$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & -2 & -7 & -1 \\ 0 & 0 & 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array} \right)$$

free variables: $x_2 = a$, $x_5 = b$.

use of equations: $x_1 = -1 - 2a + b$, $x_3 = 1 + 2b$, $x_4 = -3b$.

vector form (in \mathbf{R}^5): $(x_1, x_2, x_3, x_4, x_5) = (-1 - 2a + b, a, 1 + 2b, -3b, b)$.

2. Subtract multiples of the first equation from the second and third equation, to obtain new equations

$$-x_2 - 5x_3 = b - 3a, \quad -x_3 = c - 2a.$$

Then from the fourth equation, $x_2 + 4x_3 = d$, add the new second equation, and subtract the new third equation, to obtain

$0 = (b - 3a + d) - (c - 2a) = b - a - c + d$ as the condition to make the system consistent.

3. $(A|I) = \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 3 & 9 & 17 & 0 & 1 & 0 \\ 2 & 7 & 13 & 0 & 0 & 1 \end{array} \right)$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 & 1 \end{array} \right) \quad (r_2 \rightarrow r_2 - 3r_1, r_3 \rightarrow r_3 - 2r_1)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 7 & 0 & -3 \\ 0 & 1 & 3 & -2 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & 0 \end{array} \right)$$

Then $A^{-1} = \left(\begin{array}{ccc} 1 & 2 & -3 \\ \frac{5}{2} & -\frac{3}{2} & 1 \\ -\frac{3}{2} & \frac{1}{2} & 0 \end{array} \right)$.

4. USE ROW OPERATIONS to Reduce $A = \begin{pmatrix} 2 & 5 & 4 & 11 & 7 \\ 0 & 0 & 2 & 6 & 0 \\ 5 & 2 & 3 & 10 & 7 \\ 3 & 3 & 3 & 9 & 2 \\ 1 & 6 & 1 & 3 & 4 \end{pmatrix}$ and use the

reduction to find $\det(A)$.

Subtract multiples of the last row to obtain a matrix with 0's in the first column for the first four rows. Expand by minors on the first column to obtain

$$\det(A) = \det \begin{pmatrix} -7 & 2 & 5 & -1 \\ 0 & 2 & 6 & 0 \\ -28 & -2 & -5 & -13 \\ -15 & 0 & 0 & -10 \end{pmatrix}.$$

Subtract the second row from the first row, and add the second row to the third row, to obtain a matrix with the 2 as the entry in the second row, second column, and all other entries in the second column 0s. Expand the resulting 4-by-4 matrix on the second column, to obtain

$$\det(A) = 2 \det \begin{pmatrix} -7 & -1 & -1 \\ -28 & 1 & -13 \\ -15 & 0 & -10 \end{pmatrix} = 280.$$

5. Determine whether $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 13 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 23 \end{pmatrix} \right\}$ spans \mathbf{R}^3 .

Write the column vectors as the columns of a 3-by-4 matrix A that is the coefficient matrix of the systems $A\vec{c} = \vec{x}$, for \vec{x} in \mathbf{R}^3 that must be consistent. Reduce A , and observe that the reduced matrix has a row of 0s. Conclusion: the vectors do NOT span.

[Remark: Now that we have our conclusion, we know that there must have been relations of linear dependence among the given four vectors. We check our work by using the 0 in the first entry and 1 in the second entry of the second vector \vec{v}_2 to clear the second entry of the other three vectors. We get three vectors that are

multiples of $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. So the span of the four vectors is the same as the span

of \vec{w}, \vec{v}_2 . Next, we observe the linear combination: $a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ has

$c = -a + 3b$. For example $a = b = 1$ gives that we must have $c = 2$ for a vector with $a = b = 1$ to be in $\text{Span}(\vec{w}, \vec{v}_2)$. We confirm that the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in \mathbf{R}^3 is

not in the span of the given four vectors.]

6. Determine whether the given three vectors in \mathbf{R}^4 are linearly independent.

We again build the matrix A having the vectors as columns. We get

$$A = (\vec{v}_1, \vec{v}_2, \vec{v}_3) \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -6 & 6 \\ 0 & -8 & 8 \end{pmatrix}.$$

We see that A has rank two, so that the equation $A\vec{c} = \vec{0}$ has solutions with $\vec{c} \neq \vec{0}$, and the vectors are linearly dependent. [Explicitly, the entries c_1, c_2, c_3 of \vec{c} satisfy $c_2 = c_3$ and $c_1 + 2c_2 = 0$, so $c_3 = 1$ gives $-2\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$ as a nontrivial solution (“relation of linear dependence”).]

7. Determine whether the given three vectors in \mathbf{R}^3 are a basis.

We again build the matrix A having the vectors as columns. This is an easier case, we just check that $\det(A) \neq 0$ (in fact, $\det(A) = 1$). This assures that $A\vec{c} = \vec{0}$ has the unique solution $\vec{c} = A^{-1}\vec{0} = \vec{0}$, and the vectors are linearly independent. But $\dim(\mathbf{R}^3) = 3$, so any three linearly independent vectors span (as in Section 2.4), so the vectors are a basis.

8. omitted

Remarks on Exam 1 Fall 08

Problems 1-3 and 5-7 are similar to the corresponding problems above.

For Problem 4, the condition in the third coordinate is nonhomogeneous, $3x + 2y + 1$. When we add two vectors in this collection we get $+2$, so the collection is NOT a subspace, since it's not closed under vector addition. Alternatively, the collection W does not contain $\vec{0}$, since to get the first two coordinates equal 0, we need $x = y = 0$, but then the third coordinate is nonzero. In particular, W is (also) not closed under scalar multiplication (by the scalar 0, in particular).

For Problem 8 (on the Fall exam), review the material on coordinates at the end of 2.3. Also, note that the syllabus this semester includes 2.4 and 2.5 (not included on the syllabus for either 08 exam). That affects the solution above for Problem 7, Spring 08; as well as Wronskians from 2.5, which you should also review. For additional practice, solving the problems that appear on BOTH samples, from both exams, seems to be a good use of study time. While you're working Problem 2 on the Fall 08 exam, note that this is a nullspace problem, for which we covered additional material in 2.4