Math 205 Final Exam December 17, 2002

NAME : ____ Section _______
(Last, First)

Prof. Dodson ... (MWF 9am, Sect. 10) ___

Prof. Zhang ...... (MWF 2pm, Sect. 11) ___

Prof. Szczepanski (TuTh 8^* am, Sect. 12) ___

Prof. D'Arcy ... (TuTh 11^* am, Sect. 13) ___

Question 1: ____________ /10

Question 2: ____________ /10

Question 3: ____________ /10

Question 4: ____________ /5

Question 5: ____________ /45

Question 6 : ____________ /15

Question 7: ____________ /15

Question 8: ____________ /15

Question 9: ____________ /10

Question 10: ____________ /15

Question 11: ____________ /25

Question 12: ____________ /25

INSTRUCTIONS : This is a 3 hour exam. You are not to use a calculator during the exam. Since you will be graded on your work, be sure to indicate clearly the steps in your solution. Solutions to differential equations are not to include integrals; and must be expressed only in terms of real numbers and real functions.

$(8^* = 7:55, 11^* = 10:45)$

1. (7 *points*) (a) Solve the initial value problem $y' = y^2 \cos x,\ y(0) = 3$. Give an explicit formula, $y = y(x)$ for your solution.

(b) (3 *points*) Solve the initial value problem with the same differential equation as in part (a), but with initial value $y(0) = 0$.

2. (10 *points*) Solve the differential equation $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right) y = \dfrac{1}{x^3 + 1}$.

3. (10 *points*) Find all solutions of the following system of equations.

$$\begin{aligned} x_1 + 2x_2 - \ 3x_3 &= \ \ 1 \\ -2x_1 + \qquad 2x_3 &= -6 \\ x_1 + 5x_2 - 6x_3 &= -2 \end{aligned}$$

4. (5 *points*) Find all solutions of the following system of equations.

$$\begin{aligned} x_1 + 2x_2 - \ 3x_3 &= \ \ 1 \\ -2x_1 + \qquad 2x_3 &= -5 \\ x_1 + 5x_2 - 6x_3 &= -2 \end{aligned}$$

5. Find the general solution of the following differential equations. Your answer must be expressed in terms of real numbers and real functions (not complex numbers).

(a) (10 *points*) $y''(x) + 3y'(x) + 3y(x) = 0$

(b) (10 *points*) $(D - 2)^3(D^2 + 4)y = 0.$

(c) (10 *points*) $y''(x) + y'(x) - 2y(x) = 4e^{3x}.$

(d) (15 *points*) $y''(x) + y'(x) - 2y(x) = 4e^{-2x}.$

6. (15 *points*) (a) Find the inverse of $A = \begin{pmatrix} 2 & -1 \\ 7 & 4 \end{pmatrix}$.

(b) Use A^{-1} to find the solution of the system

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ 7x_1 + 4x_2 &= -2. \end{aligned}$$

7. (10 *points*) (a) Find all solutions of the following system of equations.

$$\begin{aligned} x_1 \qquad\quad + x_3 - 2x_4 + x_5 &= 0 \\ -x_2 + 2x_3 + x_4 \qquad\quad &= 0 \\ 3x_1 + x_2 + x_3 - 4x_4 + 3x_5 &= 0 \end{aligned}$$

(b) (5 *points*) Find a spanning set for the solutions in part (a).

8. (15 *points*) For each of the following either (i) determine that the collection is linearly dependent AND find a relation of linear dependence, or (ii) explain how you determine that the collection is linearly independent.

(a) $\mathbf{v}_1 = (1, -1, 2), \mathbf{v}_2 = (1, 1, 3)$ and $\mathbf{v}_3 = (1, -2, -2)$.

(b) $\mathbf{v}_1 = (1, -1, 2), \mathbf{v}_2 = (1, 1, 3)$ and $\mathbf{v}_3 = (1, -9, -2)$.

(c) $f_1 = \cos x$ and $f_2 = e^{5x}$.

9. (10 *points*) If S is the subspace of $\mathbf{R}^3$ consisting of all vectors of the form $(x_1, x_2, x_1 - 3x_2)$, find a basis for S. Verify that the vectors you give are linearly independent.

10. (15 *points*) Let $B = \begin{pmatrix} -2 & -1 \\ 5 & 4 \end{pmatrix}$. (a) Find the eigenvalues of B.

(b) Find all eigenvectors of B.

11. (25 *points*) You are given that $\lambda_1 = 3$ and $\lambda_2 = -1$ are the only eigenvalues of the matrix $A = \begin{pmatrix} 5 & 2 & 0 \\ -6 & -3 & 0 \\ 3 & 2 & -1 \end{pmatrix}$.

(a) Find all eigenvectors of A with eigenvalue λ_1.

(b) Find all eigenvectors of A with eigenvalue λ_2.

(c) Use your answers to (a) and (b) to decide whether A is defective or not, with a clearly stated reason supporting your decision.

12. (25 *points*) Consider the coefficient matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 2 \\ 1 & -2 & 2 \end{pmatrix}$. You are given that B has characteristic polynomial

$$p(\lambda) = (\lambda + 2)(\lambda - 1)^2.$$

Further, an eigenvector with eigenvalue $\lambda_1 = -2$ is $\vec{v}_1 = (0, 2, 1)$.

(a) Find all eigenvectors of B with eigenvalue $\lambda_2 = 1$.

(b) Use the given information and your answer in part (a) to give the general solution to $\mathbf{x}' = B\mathbf{x}$, where $\mathbf{x} = \mathbf{x}(t)$ is the vector function with entries $x_1(t), x_2(t), x_3(t)$.

(c) Use the variation-of-parameters technique ... (continued on next page)

12.(c) Use the variation-of-parameters technique to find a particular solution $\mathbf{x}_p$ to $\mathbf{x}\,'(t) = B\mathbf{x}(t) + \begin{pmatrix} 0 \\ e^{3t} \\ 0 \end{pmatrix}$, where B is the above matrix.