Math 205 Second Exam Fall 06 Solutions Spring, 2009

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1. In this problem, solutions to differential equations are not to include integrals.

(a) (10 *points*) Solve the differential equation $x\dfrac{dy}{dx} + y^2 = 0$, and write your solution as an explicit function $y = y(x)$.

(b) (10 *points*) Find the general solution of $\dfrac{dy}{dx} - \dfrac{y}{x} = 3x^3$.

Solutions:

(a) Separated equations is $-y^{-2}\,dy = x^{-1}\,dx$. Integrating,

$\dfrac{1}{y} = \ln x + c$; so the explicit solution is $y(x) = \dfrac{1}{\ln x + c}$.

(b) The integrating factor is $f = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln(x^{-1}} = \frac{1}{x}$.

Multiplying by the integrating factor, and re-writing the left side, $(\frac{y}{x})' = 3x^2$. Integrating, $\frac{y}{x} = x^3 + c$, so $y = x^4 + cx$.

2. (15 *points*) A tank whose volumn is 50 L initially contains 25 g of salt dissolved in 25 L of water. A solution containing 5 g/L of salt is pumped into the tank at a rate of 3 L/min, and the well-stirred mixture flows out at a rate of 1 L/min. Find the amount of salt in the tank after 12 minutes. Include in your solution a clear statement of the differential equation satisfied by the amount of salt $A(t)$ in the tank at time t.

Solution: The volume at time t is $V(t) = 2t + 25$.

The differentail equation is obtained from the rule "rate-in - rate-out" as

$A' = r_1c_1 - r_2c_2 = 15 - \frac{1}{V}A$,

with $r_1 = 3, c_1 = 5, r_2 = 1$ and $c_2 = \frac{A}{V}$. Using our solution for $V(t)$,

the differential equation is $A' + \frac{1}{2t+25}A = 15$.

The integrating factor here is $(2t+25)^{\frac{1}{2}}$. Then

$(A(2t+25)^{\frac{1}{2}})' = 15(2t+25)^{\frac{1}{2}}$ gives $A(t) = 5(2t+25) + \dfrac{c}{(2t+25)^{\frac{1}{2}}}$. The initial condition $A(0) = 25$ gives $c = -500$, so $A(t) = 5(2t+25) - \frac{500}{(2t+25)^{\frac{1}{2}}}$. Finally,

$A(12) = (5)(49) - \frac{500}{\sqrt{49}} = 245 - \frac{500}{7}$.

3. (25 *points*) Find the general solution of the following differential equations. Partial credit in parts (b) and (c) will be given for a clearly stated solution to the homogeneous equation. In this problem solutions to differential equations must be expressed only in terms of real numbers and real functions.

(a) $y''(t) + y'(t) - 6y(t) = 0.$

(b) $y''(t) + y'(t) - 2y(t) = -12e^{-2t}.$

(c) $y''(x) + 4y'(x) + 4y(x) = 3\cos(2x).$

Solutions:

(a) $r^2 + r - 6 = (r+3)(r-2)$, so $r = 2, -3$ (or use the quadratic formula). So $y = c_1e^{2x} + c_2e^{-3x}$.

(b) Checking the right side, we consider the trial function $y_p = Ae^{-2t}$. We first check y_H, and find $y_H = c_1e^x + c_2e^{-2t}$.

Since e^{-2t} occurs in y_H, the function Ae^{-2t} cannot possibly satisfy the non-homogeneous equation. So we take the "modified trial function"

$$y_p = Ate^{-2t}.$$

Differentiating twice, we find $-3A = -12$, so $y_p = 4te^{-2t}$, and
$y = y_H + y_p = c_1e^x + c_2e^{-2t} + 4te^{-2t}$.

(c) The solution of the homogeneous equation is $y_H = (c_1 + c_2x)e^{-2x}$. For the trial function, we take the usual trial function, $y_p = A\cos(2x) + B\sin(2x)$.

Differentiating twice, and plugging y_p' and y_p'' into the

differential equation, we equate coefficients of $\cos(2x)$ and $\sin(2x)$.

We start with $\sin(2x)$, which is the simpler coefficient, $-4B - 8A + 4B = 0$,

which gives $A = 0$. Then the coefficients of $\cos(2x)$ are $8B = 3$,

using $A = 0$. We double-check, and see $y_p = \frac{3}{8}\sin(2x)$ is a solution,

and $y = y_H + y_p = (c_1 + c_2x)e^{-2x} + \frac{3}{8}\sin(2x)$.

4. (10 *points*) Recall that the motion $y(x)$ of a spring-mass system is governed by

$$my''(x) + c\,y'(x) + k\,y(x) = 0,$$

where m is the mass, c is the friction constant and k is the spring constant. If $m = 2, c = 4$ and $k = 10$ find the solution to the Initial Value problem $y(0) = 3, y'(0) = -1$. In this problem solutions to differential equations must be expressed only in terms of real numbers and real functions.

Solution: Substituting the given values, and dividing by 2,

$y'' + 2y' + 5y = 0$. So the roots satisfy $r^2 + 2r + 5$, and $r = -1 \pm 2i$.

So $y = e^{-x} \left(c_1 \cos(2x) + c_2 \sin(2x)\right)$.

5. (15 *points*) You are given that the characteristic polynomial of the matrix $A = \begin{pmatrix} 2 & 2 & -2 \\ 0 & -2 & 0 \\ -2 & -1 & -1 \end{pmatrix}$ is $p(\lambda) = (\lambda + 2)^2(\lambda - 3)$.

(a) Find an eigenvector with eigenvalue $\lambda_2 = 3$.

(b) Give a diagonal matrix D and a matrix P so that $A = PDP^{-1}$.

Solutions:

(a) We row reduce $A - 3I$, and find that $y = 0$, that $z = a$ is a free variable, and that $x = -2a$. So an eigenvector (with $a = 1$) is $(-2, 0, 1)$.

(b) We need a basis for the eigenspace of A for $\lambda_1 = -2$.

$A+2I$ already has a 0-row, and the two nonzero rows are multiples of $(2, -1, -1)$. Both y and z are free variables, and setting

$y = 2a, z = 2b$ gives $x = a + b$. Then $(x, y, z) = (a + b, 2a, 2b) = a(1, 2, 0) + b(1, 0, 2)$,

so $(1, 2, 0)$ and $(1, 0, 2)$ are a basis. For D we take $diag(-2, -2, 3)$,

then $P = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, where we've written the eigenvectors as columns.

6. (15 *points*) You are given that the matrix $B = \begin{pmatrix} 0 & 3 \\ -3 & 6 \end{pmatrix}$ has characteristic polynomial $P(\lambda) = (\lambda - 3)^2$.

(a) Find an eigenvector with eigenvalue $\lambda = 3$.

(b) (Spring 2009 revision) Determine whether B is diagonalizable or not, with a clear reason.

Solution:

(a & b) $B - 3I = \begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

which has a one dimensional solution, so B has only one eigenvalue, and only one linearly independent eigenvector, and is not diagonalizable, since it needs two LI eigenvectors.

The free variable is $y = a$, so $x = a$, and $(1, 1)$ is an eigenvector.

Alternatively, the eigenvalue 3 has algebraic multiplicity 2, and geometric multiplicity 1, with $d_1 \neq m_1$. And a third solution for part (b) $d_1 = 1 < n = 2$, with no second distinct eigenvalue to give a second dimension d_2.