

**Week 3 (continued):**  $A\vec{x} = \vec{0}$ , homog.; Matrix Inverse.

The **Rank** of a matrix is defined to be

the number of non-zero rows in the RREF of the matrix. We do not necessarily need the RREF or even a REF to find the rank; for example, a (square) upper-triangular matrix that is  $n \times n$  with non-zero diagonal entries always has rank  $n$ . (why?)

We will discuss the results of the qualitative

theory more later on the text; but the first main result is that when  $A\vec{x} = \vec{b}$ , with  $A$  an  $m \times n$  matrix, has  $\text{rank}(A) = \text{rank}(A^\#) = n$ , the system has a unique solution.

(We're using  $A^\# = (A|\vec{b})$  for the augmented matrix.)

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Next, whenever  $\text{rank}(A) = \text{rank}(A^\#)$  we may

read-off a particular solution  $\vec{x}_p$ , so

that the system is consistent; while

$\text{rank}(A) \neq \text{rank}(A^\#)$  occurs only

when  $\text{rank}(A^\#) = \text{rank}(A) + 1$ , in which

case the last equation reads  $0 = 1$ , which

is inconsistent, so the system has no solution.

Finally, if  $r = \text{rank}(A) = \text{rank}(A^\#) < n$ , there are infinitely many solutions; and with  $r$  leading 1's; so  $r$  bound variables, there are  $d = n - r$  free variables and  $d$  linearly independent solutions to the homog. equation.

We have one more case for matrix multiplication,

3. (a)  $m \times n$  matrix  $A$  by  $q \times p$ -matrix  $B$ ,  
only when  $n = q$ , is the  $m \times p$  matrix with columns

$$\begin{aligned} AB &= A \left( \vec{b}_1, \dots, \vec{b}_p \right) \\ &= \left( A\vec{b}_1, \dots, A\vec{b}_p \right). \end{aligned}$$

3. (b) An alternate description giving the  $i, j$ th entry of  $AB$  : (( $i$ th row of  $A$ )  $\cdot$  ( $j$ th column of  $B$ ))

**Linear Combination property** of matrix mult:

we have another version of the 2nd case, above.

If  $\vec{c}$  is the column vector with entries  $c_1, c_2, \dots, c_n$ , we can also write the matrix product using columns of  $A$  as  $A\vec{c} = (\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n) \vec{c}$   
 $= c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n$ . (why?)

Any sum of scalar multiples  $x_1\vec{a}_1 + \dots + x_n\vec{a}_n$

is called a **linear combination** of the vectors

$$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n.$$

A square matrix is said to be a **diagonal**

matrix if the only non-zero entries are

on the main diagonal (top-left to lower-right).

The  $n$ -by- $n$  **identity matrix**  $I = I_n$   
 is the diagonal matrix with all diagonal  
 entries 1's.

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If  $A$  is any  $n$ -by- $n$  matrix,  
 we have  $A \cdot I = I \cdot A = A$ , so  $I$   
 behaves like the number 1 under multiplication  
 (that is,  $1a = a1 = a$ , for all real  $a$ ).

Finally,  $A$  is called **invertible** or  
**non-singular** if the matrix equation  
 $AX = XA = I$ , has an  $n$ -by- $n$  solution  $X$ , in  
 which case we write  $X = A^{-1}$ , and call  $X$   
 the **inverse** of  $A$ .

An essential property of the inverse is that  
 when it exists, it is unique. If we translate  
 the matrix equation  $AX = I$  into  $n$  systems  
 of equations for the columns of  $X = (\vec{x}_1 \dots \vec{x}_n)$ ,  
 we get  $AX = A(\vec{x}_1 \dots \vec{x}_n) = (A\vec{x}_1 \dots A\vec{x}_n) = (\vec{e}_1, \dots, \vec{e}_n)$ ,  
 $[(A\vec{x}_1 \dots A\vec{x}_n) = (\vec{e}_1, \dots, \vec{e}_n)]$  where  $\vec{e}_j$  is the  $j$ th column  
 of the identity matrix, so  $A\vec{x}_j = \vec{e}_j$ .

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Since the inverse is unique, the system for each column  
 has a unique solution, so  $A$  has rank  $n$  and the RREF of  $A$  is  $I$ .  
 Applying row reduction to the “augmented” matrix  $(A|I)$ ,  
 we get  $(I|X)$ , with  $X = A^{-1}$ .

For an example of a 3-by-3 inverse, we compute the inverse

of the matrix in **Example**  $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{pmatrix}$ .

**Solution:**

$$(A|I) = \left( \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right) \quad (r_2 \rightarrow r_2 - 2r_1, r_3 \rightarrow r_3 - r_1)$$


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$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right) \quad (r_3 \rightarrow -r_3)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 2 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right) \quad (r_2 \rightarrow r_2 + r_3, r_1 \rightarrow r_1 - 2r_3)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right) \quad (r_2 \rightarrow -r_2)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right) \quad (r_1 \rightarrow r_1 + r_2), \text{ so } A^{-1} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$