

Homework 3:

Chapter 2, Sections 4, 5 [**UPDATE**]

Due Monday, Feb 7

Section 2.4: Row Reduction
(Monday)

2.5: Gaussian Elimination
(moved to Friday)

2.6 Inverses (moved to next week)

Linear Systems

If A is an $m \times n$ matrix with entries $a_{i,j}$,

\vec{x} is the n -column vector with entries x_1, \dots, x_n ,

and \vec{b} is the m -column vector with entries b_1, \dots, b_m ,

the matrix equation $A\vec{x} = \vec{b}$ gives m equations,

each of the form (i th row of A) $\cdot \vec{x} = b_i$,

which is called a **linear system of m equations in n variables**.

In the equation $A\vec{x} = \vec{b}$, the matrix A is

called the **coefficient matrix** of the system,

and \vec{x} and \vec{b} are called the vector of

unknowns and the right-hand side vector,

respectively.

Our preliminary qualitative description of the solutions of these systems (in 2-space and 3-space) suggests three cases: (1) just one unique solution; (2) no solutions; or (3) infinitely many solns.

We say that the system of equations is **consistent** if there is at least one solution; and **inconsistent** if there are no solutions.

Two systems with the same solutions are called **equivalent**.

Our objective is to replace a given system by the simplest possible equivalent system. For the calculations - we don't use the equations, but instead one more matrix $A^\# = (A|\vec{b})$, which is called the **augmented matrix** of the system.

Problem 3.

For the system

$$\begin{aligned}x + y + z - w &= 3, \\2x + 4y - 3z + 7w &= 2\end{aligned}$$

determine the coef. matrix A , the right-hand side vector \vec{b} and the augmented matrix $A^\#$.

Solve the system using elementary row operations, and write the solution in vector form.

Solution:

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -3 & 7 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$A^\# = \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 2 & 4 & -3 & 7 & 2 \end{array} \right).$$

$$(A|\vec{b}) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 2 & -5 & 9 & -4 \end{array} \right) \quad (r_2 \rightarrow r_2 - 2(r_1))$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 1 & -\frac{5}{2} & \frac{9}{2} & -2 \end{array} \right) \quad (r_2 \rightarrow \frac{1}{2}r_2)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & \frac{7}{2} & -\frac{11}{2} & 5 \\ 0 & 1 & -\frac{5}{2} & \frac{9}{2} & -2 \end{array} \right) \quad (r_1 \rightarrow r_1 - r_2)$$

The augmented matrix has been “reduced” to

Reduced Row Echelon Form (RREF). For the last step, we notice “leading 1’s” in column 1 and 2, so we call the variables corresponding to those columns bound variables - x and y are bound. The other variables are free variables, and we use each row to solve for one bound variable.

$$x + \frac{7}{2}z - \frac{11}{2}w = 5, \text{ so}$$

$$x = 5 - \frac{7}{2}z + \frac{11}{2}w \text{ and}$$

$$y - \frac{5}{2}z + \frac{9}{2}w = -2, \text{ so}$$

$$y = -2 + \frac{5}{2}z - \frac{9}{2}w.$$

Finally, we replace the free variables by parameters,

$$z = r, w = s, \text{ then } (x, y, z, w) =$$

$$\left(5 - \frac{7}{2}r + \frac{11}{2}s, -2 + \frac{5}{2}r - \frac{9}{2}s, r, s\right)$$

$$= (5, -2, 0, 0) + r\left(-\frac{7}{2}, \frac{5}{2}, 1, 0\right) + s\left(\frac{11}{2}, -\frac{9}{2}, 0, 1\right).$$

1.1 Gauss-Jordan Elimination

We simplify the information in a matrix by using Elementary Row Operations (ERO's). There are 3:

1. $P_{ij} : r_i \leftrightarrow r_j$ means switch rows i and j .
2. $M_i(k) : r_i \rightarrow kr_i$ means multiply row i by $k \neq 0$.
3. $A_{i,j}(k) : r_i \rightarrow r_i, r_j \rightarrow (r_j + kr_i)$ means replace (row j) by the (linear) combination (row j) + k (row i), leaving row i unchanged.

We say that matrices A and B are **row equivalent** if there is a sequence of ERO's that starts with A and ends with B .

We say that a matrix is (row) reduced to simplest form if we have a matrix in reduced row echelon form (RREF) that is row equivalent. We may record the specific row operations, in the order used, that give the row reduction. The **Rank** of a matrix is the number of non-zero rows in a row echelon form of the matrix.

Problem 4. Use Gauss-Jordan Elimination to solve

$$2x_1 - x_2 + 3x_3 - x_4 = 3$$

$$3x_1 + 2x_2 + x_3 - 5x_4 = -6$$

$$x_1 - 2x_2 + 3x_3 + x_4 = 6$$

Solution: We use the augmented matrix $A^\#$,

$$\text{with } A^\# = (A|\vec{b}) \rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 3 & 2 & 1 & -5 & -6 \\ 2 & -1 & 3 & -1 & 3 \end{array} \right),$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 0 & 8 & -8 & -8 & -24 \\ 0 & 3 & -3 & -3 & -9 \end{array} \right),$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \text{ (REF),}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \text{ (RREF).}$$

[to be continued ...]

Note 1: the matrix in the next-to-last step is in Row Echelon Form (REF), but not reduced. In Gaussian elimination we may stop at a REF and continue solving using back-substitution. For Gauss-Jordan elimination, the back-substitution steps are also done as row operations on the augmented matrix. Small systems with a unique solution may be done by Gaussian elimination without confusion, since the objective is to find numbers; Gauss-Jordan elimination will be the method-of-choice for finding generators when there are infinitely many solutions, and can always be used.

Last step in Gauss-Jordan: From the REF we identify **free variables** and **bound variables**.

Again x_1 and x_2 , are bound, the other variables are free variables, and

recalling

$$(A|\vec{b})_R = \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

the 1st row corresponds to the equation

$x_1 + x_3 - x_4 = 0$, which we solve for

$x_1 = -x_3 + x_4$ and the 2nd row gives

$x_2 - x_3 - x_4 = -3$, so $x_2 = -3 + x_3 + x_4$,

and $(x_1, x_2, x_3, x_4) = (-x_3 + x_4, -3 + x_3 + x_4, x_3, x_4)$.