1. Sample exam, #5 from Fall 2006: The augmented matrix has

$\begin{bmatrix} 1 & -1 & -1 & 2 & | & 1 \\ 0 & 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$ as a row echelon matrix (not unique), and

$\begin{bmatrix} 1 & -1 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$ as as its (unique) reduced row echelon matrix.

We set $x = r$ and $z = s$ as parameters for the free variables, then $w - x + z = 2$ gives $w = 2 + r - s$ and $y - z = 1$ gives $y = 1 + s$ for the leading variables, so the solution is $[w, x, y, z] = [2 + r - s, r, 1 + s, s]$.

2. Sample exam, #7 from Fall 2006: To decide whether $\vec{v}$ is in the span of $\vec{v}_1$ and $\vec{v}_2$, we use the augmented matrix $[\vec{v}_1 \vec{v}_2 | \vec{v}]$ (see week 5 slides or Text Example 2.18, p. 90-91). We have

$$\begin{bmatrix} 1 & 1 & | & 3 \\ 1 & 0 & | & 2 \\ 2 & 1 & | & 5 \\ 3 & 2 & | & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \\ 0 & 1 & | & 1 \\ 0 & 2 & | & 0 \end{bmatrix}.$$

We halt the row reduction, and observe that the new 2nd equation and the new 4th equation are inconsistent: $c_2 = 1$ and $2c_2 = 0$. So $\vec{v}$ is NOT in the span of $\vec{v}_1$ and $\vec{v}_2$.

We could also have answered this question directly. In the vector equation $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, the second component gives $2 = c_1(1) + c_2(0) = c_1$. Plugging this value in the equations using the 1st and 4th components gives the inconsistency.

3. Sample exam, #8 from Fall 2006: To decide whether $\vec{v}_1$ and $\vec{v}_2$ and $\vec{v}_3$ are linearly independent, we use the augmented matrix $[\vec{v}_1 \vec{v}_2 \vec{v}_3 | \vec{0}]$ (see Text Example 2.23, as discussed in class for the solution of 2.3 #24 to illustrate the method for suggested problems 2.3 #23 and #28). We have the same vectors as in the previous problem, and the same reduction gives

$[\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\,] \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, which we augment by $\vec{0}$. We see that each of the variables c_1, c_2 and c_3 are leading variables, and that there are no free variables, so the solution is unique. So in the vector equation

$\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{v}_3$, $c_1 = c_2 = c_3 = 0$ is the only solution.

So the vectors ARE linearly independent.

There's a general fact here that relates these two problems. If the vectors had been linearly dependent, then an explicit relation of dependence would have given one as a linear combination of the others, and $\vec{v} = \vec{v}_3$ would have been in the span of $\vec{v}_1$ and $\vec{v}_2$.

That is, NO for #7 gives YES for #8.

4. Review problem from 2007 suggested homework, 1.3 #24. We're looking for a plane parallel to the plane $6x - y + 2z = 3$. Planes are parallel if they have the same normal vectors, so we use the normal vector of the given plane $\vec{n} = [6, -1, 2]$. Next, the plane we're looking for goes through the point $P = (0, -2, 5)$, with position vector $\vec{p} = [0, -2, 5]$. For a point $X = (x, y, z)$ with vector $\vec{x} = [x, y, z]$, we compute $\vec{n} \cdot \vec{x} = 6x - y + 2z$ and $\vec{n} \cdot \vec{p} = 0 + 2 + 10 = 12$, so the normal equation $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ gives $6x - y + 2z = 12$.

5. Additional practice, 2.2 #27. The reduced row echelon matrix of the augmented matrix is $\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$. We take the free variable $x_3 = a$, then $x_1 = -x_3 = -a$, and $x_2 = -x_3 = -a$. Our solution is $[x_1, x_2, x_3] = [-a, -a, a] = a[-1, -1, 1]$. We see that every solution is in the span of $[-1, -1, 1]$.

6. Additional practice, 2.2 #40. We were given that the system reduces to $\begin{bmatrix} 2 & -4 & | & -6 \\ 0 & 2+2k & | & 3+3k \end{bmatrix}$. If $2 + 2k \neq 0$, then both x and y are leading variables, there is no $\vec{0}$-row in the reduction of the coefficient matrix, and no free variable, so the system is consistent and unique. If $2 + 2k = 0$, then $k = -1$, and the last row of the augmented matrix is $[00|0]$. Again the system is consistent, but y is a free variable, so there are infinitely many solutions. These two cases include all values of k, and there is none that gives an inconsistent system.