Math 205, Fall 2010

Homework 14: due Wed, Dec. 8th

Chapter 7, Section 6, plus problems 8,9 from Course Site.

Week 14: non-homog systems

Recall that if A is n-by-n, and

if $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ are LI

solutions of the homog equation $\vec{x}' = A\vec{x}$,

the fundamental matrix is $X = (\, \vec{x}_1(t) \ \ldots \ \vec{x}_n(t) \,)$.

To solve the non-homog VDE

$\vec{x}' = A\vec{x} + \vec{b}$, we solve

for $\vec{x}_p = X\vec{u}$ using $X\vec{u}' = \vec{b}$.

As a last example,

Problem 7.6.

Solve $\vec{x}' = A\vec{x} + \vec{b}$

when $A = \begin{pmatrix} -1 & -2 & 2 \\ 2 & 4 & -1 \\ 0 & 0 & 3 \end{pmatrix}, \vec{b} = \begin{pmatrix} -e^{3t} \\ 4e^{3t} \\ 3e^{3t} \end{pmatrix}$

Solution.

$P(\lambda) = -\lambda(\lambda - 3)^2$;

for $\lambda_1 = 0, \vec{v}_1 = (-2, 1, 0)$; and

for $\lambda_2 = 3, A - 3I \to \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

so a basis for the eigenspace is

$\vec{v}_2 = (-1, 2, 0)$ and $\vec{v}_3 = (1, 0, 2)$; and

$X = \begin{pmatrix} -2 & -e^{3t} & e^{3t} \\ 1 & 2e^{3t} & 0 \\ 0 & 0 & 2e^{3t} \end{pmatrix}$ is a fundamental matrix.

We solve $X\vec{u}' = b$ using row reduction

on the augmented matrix $X^{\#} = (X|\vec{b})$

$$\to \left(\begin{array}{ccc|c} -2 & -e^{3t} & 0 & -\frac{5}{2}e^{3t} \\ 1 & 2e^{3t} & 0 & 4e^{3t} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2e^{3t} & 0 & 4e^{3t} \\ 0 & 3e^{3t} & 0 & \frac{11}{2}e^{3t} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3}e^{3t} \\ 0 & 1 & 0 & \frac{11}{6} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right).$$

So $u_1 = \frac{1}{9}e^{3t}, u_2 = \frac{11}{6}t, u_3 = \frac{3}{2}t$, and

$$x_p = \tfrac{1}{9}e^{3t}\begin{pmatrix}-2\\1\\0\end{pmatrix} + \tfrac{11}{6}te^{3t}\begin{pmatrix}-1\\2\\0\end{pmatrix} + \tfrac{3}{2}te^{3t}\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{2}{9}e^{3t} - \frac{1}{3}te^{3t}\\ \frac{1}{9}e^{3t} + \frac{11}{3}te^{3t}\\ \frac{3}{2}te^{3t}\end{pmatrix},$$

and $\vec{x} = \vec{x}_c + \vec{x}_p$

$$= c_1\begin{pmatrix}-2\\1\\0\end{pmatrix} + c_2e^{3t}\begin{pmatrix}-1\\2\\0\end{pmatrix} + c_3e^{3t}\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$+ \begin{pmatrix}-\frac{2}{9}e^{3t} - \frac{1}{3}te^{3t}\\ \frac{1}{9}e^{3t} + \frac{11}{3}te^{3t}\\ \frac{3}{2}te^{3t}\end{pmatrix}.$$