Week 5b

Chapter 6, Sections 5 and 6;

Chapter 7, Sections 1, 4, 6

Week 5b:

6.5 Springs

6.6 Circuits

7.1 First Order Systems of DE

7.4 Nondefective Systems (Homogeneous)

7.6 Nonhomogeneous Systems

The general homogeneous mass-spring system is

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0,$$

where m is the mass, k is the spring constant and c is the damping constant (also called "friction"). In the case $c = 0$ the system is said to be **undamped.**

When $c \neq 0$ the system is **damped,** and the motion depends primarily on the type of damping, determined by the roots.

Problem 5 Solve $y'' + 2y' + 5y = 0$, with IV $y(0) = 1, y'(0) = 3$, and determine whether the system is **overdamped, critically damped** or **damped and oscillating,** describe the sketch.

(soln.) char. polyn. $r^2 + 2r + 5 = 0$, roots are $-1 \pm 2i$, complex, so the system is underdamped. With $y = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$ we get $1 = y(0) = c_1$,

$y = e^{-t}(\cos 2t + c_2 \sin 2t)$, so

$y' = -e^{-t}(\cos 2t + c_2 \sin 2t) + e^{-t}(-2 \sin 2t + 2c_2 \cos 2t))$,

(continued ...)

$3 = y'(0) = -(1 + 0) + (0 + 2c_2)$, so $c_2 = 2$,

$y = e^{-t}(\cos 2t + 2 \sin 2t)$.

(graphing observations: see 3 early max/mins,

2 equilibrium crosses, exponential decay.)

Problem 6:

Solve $y'' + 3y' + 2y = 3\cos t, y(0) = 2, y'(0) = -1$.

Solution:

$y_p = A_0 \cos t + B_0 \sin t$, since y_H is given by

roots $r = -1, -2$. The motion in the homog. system

is overdamped. We have $y_p' = -A_0 \sin t + B_0 \cos t$;

$y_p'' = -A_0 \cos - B_0 \sin t$,

So $3\cos t = (-A_0 \cos - B_0 \sin t) + 3(-A_0 \sin t + B_0 \cos t)$

$+2(A_0 \cos x + B_0 \sin x)$. Then $0 = -B_0 - 3A_0 + 2B_0$

gives $B_0 = 3A_0$; $3 = -A_0 + 3B_0 + 2A_0 = A_0 + 3(3A_0)$,

so $A_0 = \frac{3}{10}$, $B_0 = \frac{9}{10}$,

$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{3}{10}\cos t + \frac{9}{10}\sin t$.

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3 types, many cases:

(a) homogeneous, no damping or un-damped:

Amplitude, frequency ...

(b) homogeneous, with damping:

underdamping (complex roots, oscillating), critical damping

(repeated real), overdamping (distinct real)

(c) non-homogeneous: $y = y_H + y_p$ where (with damping)

$y_H \to 0$ as $t \to \infty$ is the **transient solution,**

and $y = y_H + y_p \to y_p$ as $t \to \infty$, so y_p is the

steady-state solution.

Finally, a case with no damping:

Problem 7 Solve $y'' + 4y = 0, y(0) = 2, y'(0) = 4$.

(Soln.) char. polynomial $r^2 + 4$, roots $\pm 2i$, general soln

$y = c_1 \cos 2x + c_2 \sin 2x$; so $2 = y(0) = c_1$,

$y = 2\cos 2x + c_2 \sin 2x$. Then

$y' = -4 \sin 2x + 2c_2 \cos 2x$, so $4 = y'(0) = 2c_2$, gives

$c_2 = 2$, and IVP soln $y = 2\cos 2x + 2\sin 2x$.