Math 205, Summer I, 2016

Week 5a:

Chapter 6, Section 1, 2, 3 and 6.5

6.1 Higher Order DE

6.2 Constant Coef, Homogeneous DE

6.3 Nonhomogeneous DE (undetermined coef)

6.5 Springs

General Solution: (1) to solve the 2nd order

linear DE $y'' + a_1 y' + a_2 y = 0$

find two linearly independent solutions y_1 and y_2,

then the general solution is $y_H = c_1 y_1 + c_2 y_2$.

(2) to solve the non-homogeneous 2nd order

linear DE $y'' + a_1 y' + a_2 y = F$,

find a particular solution y_p, then the general

solution is $y = y_H + y_p$, where y_H is the solution of (1).

Problem 1:

Determine all values r so $y = e^{rx}$ is a

solution to $y'' - 4y' + 3y = 0$.

Find the general solution.

Solution:

For $y = e^{rx}$, we have $y' = re^{rx}$ and $y'' = r^2 e^{rx}$.

Substitution in the DE gives

$0 = y'' - 4y' + 3y = r^2 e^{rx} - 4re^{rx} + 3e^{rx}$

$= (r^2 - 4r + 3)e^{rx}$.

Now $r^2 - 4r + 3 = (r-3)(r-1) = 0$ has roots $r_1 = 1$ and $r_2 = 3$, so we get solutions $y_1 = e^x$ and $y_2 = e^{3x}$, so the general solution is $y = c_1 e^x + c_2 e^{3x}$.

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Constant Coef, Homogeneous DE

When the DE $y'' + a_1 y' + a_2 y = 0$ has
coefficients a_1 and a_2 that are constant,
the two linearly independent solutions y_1 and y_2
in the general solution $y_H = c_1 y_1 + c_2 y_2$ may
be determined using the roots of the characteristic
polynomial $P(r) = r^2 + a_1 r + a_2$.

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Three cases:

(1) For distinct real roots r_1 and r_2,
$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

(2) For a repeated real root r_1 the 2nd independent
solution is $y_2 = xe^{r_1 x}$, and $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$.

(3) For a pair of complex roots $r = a + bi$ and $r = a - bi$
with a and b real, $b > 0$,
$y = c_1 e^{ax} \cos(bx) + c_2 e^{ax} \sin(bx)$.

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Recitation on 6.2.

Problem 2:
Solve $y'' - 6y' + 34y = 0$.

Solution:

The characteristic polynomial is $r^2 - 6r + 34$, so the quadratic formula gives roots $\dfrac{6 \pm \sqrt{6^2 - 4 \cdot 34}}{2}$

$= 3 \pm 5i$. (why?)

So $y = e^{3x}(c_1 \cos 5x + c_2 \sin 5x)$ is the general solution.

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***Postscript:** We may

solve Example 5 with the IVP $y'' + 4y' + 4y = 0,\ y(0) = 1, y'(0) = 4$.

We also may solve other IVP problems (time

permitting). In any case, the calculation of constants

c_1 and c_2 to solve an IVP look rather different in

(at least) four cases: the three types of roots in the

homogeneous DE, and

(fourth) the non-homogeneous DE.

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6.3 **Undetermined Coefficients**

Problem 3.

Solve $y'' - 4y' + 3y = 6e^{2x}$.

Solution:

(a) Find the general solution y_H of the

homogeneous equation $y'' - 4y' + 3y = 0$.

(a) $y_H = c_1 e^x + c_2 e^{3x}$.

(b) Determine the value of the constant A_0 so

$y_p = A_0 e^{2x}$ is a particular solution.

$y_p = A_0 e^{2x}, y_p' = A_0(2e^{2x}), y_p'' = 4A_0 e^{2x}$, so

$6e^{2x} = 4A_0 e^{2x} - 4(2A_0 e^{2x}) + 3(A_0 e^{2x})$

$= -A_0 e^{2x}$, so $A_0 = -6, y_p = -6e^{2x}$.

(c) Determine the general solution.

$y = y_H + y_p = c_1 e^x + c_2 e^{3x} - 6e^{2x}$.

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Compare 3d: If $F = 3x^2$, find constants a_0, a_1 and a_2

so $y_p = a_0 + a_1 x + a_2 x^2$.

$y_p' = a_1 + 2xa_2$, $y_p'' = 2a_2$ gives polynomial equation

$3x^2 = 2a_2 - 4(a_1 + 2xa_2) + 3(a_0 + a_1 x + a_2 x^2)$ all x.

x^2**-coef:** $3 = 3a_2$, so $a_2 = 1$.

x**-coef:** $0 = -4(2a_2) + 3a_1$ with $a_2 = 1$, so $a_1 = \frac{8}{3}$.

constant term: $0 = 2a_2 - 4(a_1) + 3(a_0)$, so $a_0 = \frac{26}{9}$.

So $y_p = \frac{26}{9} + \frac{8}{3}x + x^2$.

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We have one more **linearity property:**

If $y'' + a_1 y' + a_2 y = F_1$ and $y'' + a_1 y' + a_2 y = F_2$

have particular solutions y_{p1} and y_{p2}, then

$y'' + a_1 y' + a_2 y = F$, for $F = F_1 + F_2$, has

particular solution $y_p = y_{p1} + y_{p2}$.

Example: $y'' - 4y' + 3y = 6e^{2x} + 3x^2$ has

solution $y_p = -6e^{2x} + \frac{26}{9} + \frac{8}{3}x + x^2$.

Problem 4:

Solve $y'' + 4y' + 5y = 24 \sin x$.

Solution:

The usual trial function for $F(x) = 24 \sin x$ is $y_p = A_0 \cos x + B_0 \sin x$. Before attempting to compute y_p' and y_p'' to find A_0 and B_0 (the "undetermined coefficients"), we check y_H for repetition. The homogeneous equation $y'' + 4y' + 5y = 0$ has characteristic polynomial $P(r) = r^2 + 4r + 5$. The quadratic formula gives roots $-2 \pm i$, say $r_1 = -2 + i$ and $r_2 = -2 - i$ (why?). But the "driving force" $F(x) = 24 \sin x$ occurs as a solution of the DE with roots $\pm i$, that is, $a \pm bi$ with $a = 0, b = 1$, which do not coincide with r_1, r_2 : the **usual** $y_p = A_0 \cos x + B_0 \sin x$ is the correct choice.

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Now with $y_p = A_0 \cos x + B_0 \sin x$, we have $y_p' = -A_0 \sin x + B_0 \cos x$, and $y_p'' = -A_0 \cos x - B_0 \sin x$. Substitution in the DE gives $24 \sin x = y_p'' + 4y_p' + 5y_p$
$= (-A_0 \cos x - B_0 \sin x) + 4(-A_0 \sin x + B_0 \cos x) +$
$5(A_0 \cos x + B_0 \sin x)$.

Equating like terms

coef of $\cos x : -A_0 + 4B_0 + 5A_0 = 0$, so $B_0 = -A_0$.

coef of $\sin x : -B_0 + 4(-A_0) + 5B_0 = 24$,

so $-4A_0 + 4(B_0) = -8A_0 = 24$, gives $A_0 = -3, B_0 = 3$, then $y_p = -3 \cos x + 3 \sin x$, and

$y = y_H + y_p = e^{-2x}(c_1 \cos x + c_2 \sin x) - 3 \cos x + 3 \sin x$.

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Preliminaries on applications:

The above solution $y = y_H + y_p =$

$e^{-2t}(c_1 \cos t + c_2 \sin t) - 3\cos t + 3\sin t$

has a typical property found in our applications to springs and circuits. For physical reasons, y_H usually has an exponential decay as $t \to \infty$, so y_H is referred to as the **transient solution** and we get $y = y_H + y_p \to y_p$ as $t \to \infty$. We call y_p the **steady-state solution** because of this property.

Question: How does the behavior of y_p differ in the usual trial solution, as compared with the modified solution? (resonance)

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Problem 4b:

Solve $y'' + 16y = 24\cos(4x)$.

Solution:

There are several problems here. Taking $y_p = A\cos(4x)$ may not be enough, as $y_p' = -4A\sin(4x)$. So we might try $y_p = A\cos(4x) + B\sin(4x)$. While that would be the correct trial function for a problem like Solve $y'' + 2y' + 16y = 24\cos(4x)$, it doesn't work here. The reason: the above equation has general homog solution $y_H = c_1 \cos 4x + c_2 \sin 4x$, so for any A and B we'd have $y_H + y_p = y_H$, with no new independent solutions.

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Let's try again. **Problem 4b:**

Solve $y'' + 16y = 24\cos(4x)$.

Solution:

Take $y_p = x(A\cos 4x + B\sin 4x)$.

So $y_p' = (A\cos 4x + B\sin 4x) + x(-4A\sin 4x + 4B\cos 4x)$,

and $y_p'' =$

$= 2(-4A\sin 4x + 4B\cos 4x) + x(-16A\cos 4x - 16B\sin 4x)$,

So "equating like terms" in

$24\cos 4x = y_p'' + 16y_p$

$= [2(-4A\sin 4x + 4B\cos 4x)+$

$x(-16A\cos 4x - 16B\sin 4x)] + 16(x(A\cos 4x + B\sin 4x)$

gives $24 = 8B, 0 = -8A$. So the above $y_p = 3x\sin(4x)$.

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Recall **General Solution:** (2) to solve a 2nd order non-homogeneous 2nd linear DE $y'' + a_1y' + a_2y = F$,

find a particular solution y_p, then the general solution is $y = y_H + y_p$, where y_H is the general solution of the homogeneous equation (same coef, $F = 0$).

Also recall that we have a collection of functions that occur as solutions of (second order?) homogeneous DE:

(1) e^{ax}; (2) xe^{ax}; (3) x^2 (for $a = 0, y''' = 0$);

or, (4) $e^{ax}\cos(bx)$ or $e^{ax}\sin(bx)$.