Week 4b:

Chapter 5, Sections 6, 7 and 8 (5.5 is NOT on the syllabus)

5.6 Eigenvalues and Eigenvectors
5.7 Eigenspaces, nondefective matrices

5.8 Diagonalization [*** See next slide ***]

A vector $\vec{v} \neq \vec{0}$ in $\mathbf{R}^n$ (or in $\mathbf{C}^n$)

is an **eigenvector with eigenvalue** λ of

an n-by-n matrix A if $A\vec{v} = \lambda\vec{v}$.

We re-write the vector equation as $(A - \lambda I_n)\vec{v} = \vec{0}$,

which is a homogeneous system with coef

matrix $(A - \lambda I)$, and we want λ

so that the system has a non-trivial solution.

We see that the eigenvalues are the roots

of the **characteristic polynomial** $P(\lambda) = 0$,

where $P(\lambda) = \det(A - \lambda I)$.

To find the eigenvectors we find the distinct

roots $\lambda = \lambda_i$, and for each i

solve $(A - \lambda_i I)\vec{v} = \vec{0}$.

Problem 1. Find the eigenvalues and eigenvectors

of $A = \begin{pmatrix} 3 & -1 \\ -5 & -1 \end{pmatrix}$

Solution.

$$P(\lambda) = det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & -1 \\ -5 & -1-\lambda \end{vmatrix}$$

$= (-1-\lambda)(3-\lambda) - 5 = \lambda^2 - 2\lambda - 3 - 5$

$= \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$, so $\lambda_1 = 4, \lambda_2 = -2$.

For $\lambda_1 = 4$, we reduce the coef matrix of the system $(A - 4I)\vec{x} = \vec{0}$,

$$A - 4I = \begin{pmatrix} 3-4 & -1 \\ -5 & -1-4 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -5 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so for $x_2 = a$ we get $x_1 = -a$, and the eigenvectors for $\lambda_1 = 4$ are the vectors $(x_1, x_2) = a(-1, 1)$ with $a \neq 0$, from the space with basis $(-1, 1)$.

For $\lambda_2 = -2$, we start over with coef

$$A + 2I = \begin{pmatrix} 3+2 & -1 \\ -5 & -1+2 \end{pmatrix} \to \begin{pmatrix} 5 & -1 \\ 0 & 0 \end{pmatrix}.$$

To find a spanning set without fractions, we anticipate that finding x_1 will use division by 5, and take $x_2 = 5b$. Then $5x_1 - x_2 = 0$ gives $5x_1 = 5b$, so $x_1 = b$, and $(x_1, x_2) = b(1, 5)$, spanned by $(1, 5)$ [or, if you must, $(\frac{1}{5}, 1)$].

Problem 2.

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 10 & -12 & 8 \\ 0 & 2 & 0 \\ -8 & 12 & -6 \end{pmatrix}$.

Solution.

Since A is 3-by-3, the characteristic polynomial $P(\lambda)$ is cubic, and we think strategically: How should $P(\lambda)$ be given?

For example, do we need the coef of λ^2? We're looking for the roots, so we need the factors $(\lambda - \lambda_i)$. For that objective
there is only one good method of computing the determinant;
expansion on the 2nd row. Look why the 2nd row exp is best:
we get $P(\lambda) = (2-\lambda)\begin{vmatrix} 10-\lambda & 8 \\ -8 & -6-\lambda \end{vmatrix}$
$= (2-\lambda)[(10-\lambda)(-6-\lambda)+64] = (2-\lambda)[\lambda^2 - 4\lambda - 60 + 64] = -(\lambda-2)^3.$
Can we agree that $P(\lambda) = -\lambda^3 + 12\lambda^2 - 24\lambda + 8$,
[which we would have gotten from the other methods]
is not as useful as having a factor [since the other factors only
need a quadratic; not solving a cubic]?

Anyway, this matrix only has one distinct
eigenvalue, $\lambda_1 = 2$. To find the eigenvectors,
we reduce the coef matrix $A - 2I$

$$= \begin{pmatrix} 10-2 & -12 & 8 \\ 0 & 2-2 & 0 \\ -8 & 12 & -6-2 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that x_2 and x_3 will be free variables,
and anticipating that finding x_1 will involve
division by 2, take $x_2 = 2s$ and $x_3 = t$, so
$x_1 = 3s - t$; and $(x_1, x_2, x_3) = (3s-t, 2s, t)$
$= s(3,2,0) + t(-1,0,1)$, has LI spanning vectors
$(3,2,0)$ and $(-1,0,1)$, giving eigenvectors except when $s = t = 0$.

Problem 3. Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$.

Solution.

$$P(\lambda) = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = [(3-\lambda)(-1-\lambda)+8] = \lambda^2 - 2\lambda - 3 + 8$$

$= \lambda^2 - 2\lambda + 5$, with roots $\lambda = 1 \pm 2i$. So there are no real λ for which there is a non-trivial real solution $\vec{x} \in \mathbf{R}^2$, but there is a complex solution $\vec{0} \neq \vec{x} \in \mathbf{C}^2$. We solve for $\vec{x}$ by reducing the coef matrix as usual.

An n-by-n matrix A is defined to be **nondefective** if A has n linearly independent eigenvectors. A related term is **diagonalizable.** The matrix A is diagonalizable if there is an invertible n-by-n matrix P so that $P^{-1}AP = D$ is a diagonal matrix. We will see that A is diagonalizable if and only if A has n linearly independent eigenvectors.

Problem 4.

For each eigenvalue, find the multiplicity and a basis for the eigenspace and determine whether $A = \begin{pmatrix} 4 & 1 & 6 \\ -4 & 0 & -7 \\ 0 & 0 & -3 \end{pmatrix}$ is nondefective.

Solution.

Expanding $\det(A - \lambda I)$ on the 3rd row,

we have $P(\lambda) = (-3-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ -4 & -\lambda \end{vmatrix}$,

$= -(3+\lambda)[-4\lambda + \lambda^2 + 4] = -(3+\lambda)(\lambda-2)^2$,

so $\lambda_1 = 2$ is an eigenvalue with multiplicity 2,

and $\lambda_2 = -3$ is a simple root (multiplicity 1).

For $\lambda_1 = 2$, $A - 2I = \begin{pmatrix} 2 & 1 & 6 \\ -4 & -2 & -7 \\ 0 & 0 & -5 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

We already have sufficient info to determine

that A is not diagonalizable: Rank$(A - 2I)$=2, so

the dimension of the eigenspace (nullspace!) is 3-2 =1,

less than the multiplicity. We still need bases of the eigenspaces;

and see that $\vec{v}_1 = (-1, 2, 0)$ is a basis for $\lambda_1 = 2$.

For $\lambda_2 = -3$, $A - (-3I) = A + 3I = \begin{pmatrix} 7 & 1 & 6 \\ -4 & 3 & -7 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 7 & -8 \\ 0 & -25 & 25 \\ 0 & 0 & 0 \end{pmatrix}$

(we use $R_1 \to R_1 + 2R_2$; update, then use $R_2 \to R_2 - 4R_1$)

$\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ ($R_1 \to -R_1, R_2 \to -\frac{1}{25}R_2$; update, then $R_1 \to R_1 + 7R_2$.)

So a basis is $\vec{v}_2 = (-1, 1, 1)$, $\dim(E_{-3})$=3-2=1.

Problem 5.

Determine whether $A = \begin{pmatrix} 1 & -3 & 1 \\ -1 & -1 & 1 \\ -1 & -3 & 3 \end{pmatrix}$ is diagonalizable or not,

given $P(\lambda) = (\lambda - 2)^2(\lambda + 1)$.

Solution.

Since $\lambda_2 = -1$ is a simple root it is non-defective;

and we only need to check $\lambda_1 = 2$. We have $A - 2I = \begin{pmatrix} -1 & -3 & 1 \\ -1 & -3 & 1 \\ -1 & -3 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $\dim(E_2) = 3 - 1 = 2$, and A is diagonalizable.

We're not asked for a basis here, but for practice,

$x_2 = a, x_3 = b$ and $x_1 + 3x_2 - x_3 = 0$ gives $x_1 = -3a + b$,

$(x_1, x_2, x_3) = (-3a + b, a, b) = (-3a, a, 0) + (b, 0, b) = a(-3, 1, 0) + b(1, 0, 1)$,

so a basis is $(-3, 1, 0)$ and $(1, 0, 1)$.

We collect the terminology and results used. If $P(\lambda)$ written in factored form is $P(\lambda) = (\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_r)^{m_r}$, where the λ_j are the distinct roots, we say that λ_j has (algebraic) multiplicity m_j, or that λ_j is a simple root (multiplicity 1) if $m_j = 1$.

Let $d_j = \dim(E_{\lambda_j})$ be the dimension of the eigenspace (sometimes called the geometric multiplicity), then the main facts are

1. $1 \leq d_j \leq m_j$;
2. A is non-diagonalizable exactly when there is some eigenvalue (which must be a repeated root) that is defective, $d_j < m_j$;
3. A is diagonalizable exactly when every eigenvalue is non-defective, $d_j = m_j$, for $j = 1, \dots, r$;
4. In particular, if $P(\lambda)$ has distinct roots then A is always diagonalizable $(1 \leq d_j \leq m_j = 1)$.

We recall that m_j - the algebraic multiplicity - is the number of times the jth distinct eigenvalue is a root of the characteristic polynomial; and d_j - the geometric multiplicity - is the dimension of the λ_jth eigenspace (that is, the nullspace of $A - \lambda_j I$).

[*Starting 5.8*] First, we have a new definition, matrices
A and B are **similar** if there is an n-by-n
matrix P so that P has an inverse and $B = P^{-1}AP$.

We also recall that a **diagonal matrix** $D = diag(d_1, \ldots, d_n)$
is a square matrix with all entries 0 except (possibly) on the
main diagonal, where the entries are $d_1, \ldots, d_n$ (in that order).

Finally, a matrix A is **diagonalizable** if there is a matrix
P so that $P^{-1}AP = D$ is a diagonal matrix.

Theorem. The matrix A is diagonalizable exactly when
A has n linearly independent eigenvectors.

Further, (1) the diagonal entries in the diagonalization are the eigenvalues,
each occurring as many times on the diagonal as the multiplicity m_j;

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(2) the columns of the matrix P that diagonalizes A
are n LI eigenvectors of A; and,

(3) the j-th column of P has eigenvalue that is the j-th
entry on the diagonal.

Finally,

Problem 6. Compare eigenspaces and the defective condition

$$\text{for } A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{pmatrix}.$$

Solution. Both matrices have char polyn $P(\lambda) = -(\lambda - 4)^2(\lambda + 1)$.

Since $\lambda = -1$ a simple root (not repeated), it is non-defective.

$$\text{We have } A - 4I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$A_1 - 4I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

These matrices test our eigenvector-finding-skills. For the 2nd, the equations say $x_2 = x_3 = 0$; while we must have a nontrivial solution. Do you see one? Don't Think! Our reflex is x_2, x_3 bound; $x_1 = a$ free, so $(x_1, x_2, x_3) = (a, 0, 0) = a(1, 0, 0)$. We also have $\vec{v}_1 = (1, 0, 0)$ an eigenvector for the first matrix, but there's also a 2nd LI solution, $\vec{v}_2 = (0, -3, 2)$. So the roots and multiplicities are the same, only one entry is changed, but A is diagonalizable; while A_1 is non-diagonalizable.