Math 205, Summer I, 2016

Week 3a (continued):

Chapter 4, Sections 5 and 6.

Week 3b

Chapter 4, [Sections 7], 8 and 9

4.5 Linear Dependence, Linear Independence

4.6 Bases and Dimension

4.7 Change of Basis,

4.8 Row and Column Space

4.9 Rank-Nullity Theorem

Recall that a Vector Space V may be any collection with
formulas for vector plus and scalar mult, $V = (V, +\ ,\ \cdot)$ with 10 rules.

In practice, in Math 205, Vector Spaces
will usually be subsets of one of four standard

Examples: $\mathbf{R}^n$; $V = M_{m\times n}(\mathbf{R})$;

$C^k(a, b) =$ functions: f+g, k·f, 0, -f with k derivatives

$P_n =$ polynomials with real coef. degree $< n$.

We usually assume the 10 rules for these four vector spaces
(the rules are easier to verify than to remember).
So to establish that a subset S of one of these spaces is a vector space
we only need to check the last two rules, in
which case we say that S is a
Subspace of V. The
two subspace conditions are the
closure rules (1) for every
$\vec{u}, \vec{v} \in S$, check $\vec{u} + \vec{v} \in S$ (closure under vector $+$),
and (2) for every $\vec{u} \in S$, and every scalar
$c \in \mathbf{R}$, check $c\vec{u} \in S$ (closure under scalar mult).

For the formal definition of **span** and **spanning set** we take
vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in a vector space V. For the most part,
we think of V as being (1) $\mathbf{R}^n$; (2) $C^k(a, b)$; or, (3) a subspace of
either (a) Euclidean n-space or (b) of k-differentiable functions.

A vector $\vec{v} \in V$ is called a
linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there are
scalars $c_1, c_2, \ldots, c_k$ so that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$.
The **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is the collection of all $\vec{v} \in V$ so that
$\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.
The Span is a subspace of V. If Span $= V$,
we say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is a **spanning set** of V.

Examples: (1) $\vec{i}, \vec{j}$ span R^2;

(2) $\{\vec{i}, \vec{j}, \vec{k}\}$ is a spanning set for R^3;

(3) the coef vectors obtained from the RREF of A span the solutions of $A\vec{x} = \vec{0}$.

Linear Independence

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in a

vector space V are **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_k$ so $\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$, with at least one $c_j \neq 0$. Such a vector equation is said to be a **relation** of linear dependence among the $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. A collection of vectors for which the only solution of the vector equation is the **trivial solution,** $c_1 = 0, \ldots, c_k = 0$ is said to be **linearly independent.**

Examples: (1) $\vec{i}, \vec{j}$ in R^2; and

(2) $\{\vec{i}, \vec{j}, \vec{k}\}$ in R^3; are linearly independent.

Verification: $c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (c_1, c_2, c_3)$

$= (0,0,0) = \vec{0}$, only when $c_1 = 0, c_2 = 0, c_3 = 0$.

We start with verifications of linear independence in subspaces of $\mathbf{R}^n$

Problem

Determine whether the vectors $(1,2,3)$, $(1,-1,2)$ and $(1,-4,1)$ are LI or LD in $\mathbf{R}^3$.

If they are LD find a relation of linear dependence.

Solution [again; same method!]

Write the equation as an equation in column vectors,

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}1\\-1\\2\end{pmatrix} + c_3\begin{pmatrix}1\\-4\\1\end{pmatrix}$$
$$= \begin{pmatrix}1 & 1 & 1\\2 & -1 & -4\\3 & 2 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}.$$

Observe that this is the homog linear system

with coef matrix $A = (\vec{v}_1\vec{v}_2\vec{v}_3)$.

For n-by-n systems the system has a

unique solution exactly when the determinant is non-zero.

We have $\begin{vmatrix}1 & 1 & 1\\2 & -1 & -4\\3 & 2 & 1\end{vmatrix} = \begin{vmatrix}1 & 1 & 1\\0 & -3 & -6\\0 & -1 & -2\end{vmatrix} = 0$

so the system has nontrivial solutions

and the vectors are LD. To find a

relation we continue the reduction

$A \to \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\end{pmatrix}$,

so a spanning vector is $(1, -2, 1)$,

and $\vec{0} = \vec{v}_1 - 2\vec{v}_2 + \vec{v}_3$

is a nontrivial relation of LD.

Linear Independence of functions: we use

the Wronskian.

Bases and Dimension

Our main objective is the definition of basis.

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in a vector space

V that are are both (1) a spanning set

for V and (2) linearly independent

are called a **basis** for V. So

(1) $\vec{i}, \vec{j}$ are a basis of R^2; and

(2) $\{\vec{i}, \vec{j}, \vec{k}\}$ is a basis of R^3.

A vector space has many bases, but the

main fact is that if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$

is a basis of V and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_j$

is a second basis of V, then the number of

vectors is the same: $k = j$. The number

of vectors in a basis of V is called the **dimension** of V.

We start with verifications of linear independence in subspaces of $\mathbf{R}^n$

Example: Show that the vectors $\vec{v}_1 = (1,1)$ $\vec{v}_2 = (1,-1)$ are a basis of $\mathbf{R}^2$.

Solution. We know that $\vec{i}, \vec{j}$ are a basis of $\mathbf{R}^2$; so that the dimension of $\mathbf{R}^2$ is $\ldots$.

We check that $\vec{v}_1 = (1,1)$ and $\vec{v}_2 = (1,-1)$

are linearly independent (LI) since $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

$= -2 \neq 0$. Checking that $\vec{v}_1$ and $\vec{v}_2$ span $\mathbf{R}^2$

by expressing each $\vec{v} = (x,y) \in \mathbf{R}^2$ as a

linear combination (as we did for $\vec{i}, \vec{j}$)

takes a little bit of effort, and involves fractions.

Checking that each vector equation

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ has a

solution (without finding c_1, c_2) is easier. (why?)

But the main fact has a **practical version**

that saves any computation, or even very much

further thought. If $\vec{v}_1$ and $\vec{v}_2$

did not span $\mathbf{R}^2$ there would be a 3rd vector

$\vec{v}_3$ not in the span of $\vec{v}_1, \vec{v}_2$.

But that would make $\vec{v}_1, \vec{v}_2, \vec{v}_3$

linearly independent. We could consider

adding further vectors, each new one giving a

larger collection of indep vectors; but
we already have too many, as we've shown
$\dim(\mathbf{R}^2) \geq 3$. So $\vec{v}_1, \vec{v}_2$ have to span
(for free, given the main fact),
so $\vec{v}_1, \vec{v}_2$ is a basis for $\mathbf{R}^2$.
This is the use of the text's

Theorem 4.6.10. If the $\dim(V) = n$ for a vector space V, then any LI set of n vectors is a basis of V.

Problem

Let S be the subspace of $M_2(\mathbf{R})$ consisting of all 2×2 matrices with trace 0. Find a basis for S and use that to determine the dimension of S.

Solution

We first describe S. We recall that when

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, Tr(A) = a + d.$$

So S consists of matrices with $d = -a$. Think of the entries as parameters, then

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

expresses each matrix in S as a

linear combination of the three matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so they're a spanning set. But if

a combination with coef a, b, c as

above gave $A = 0$, the zero matrix,

then $a = 0, b = 0, c = 0$, so this spanning

set is indep, therefore a basis, and

the dimension of S is three.