Week 2a:

1. **Homework 5:**

   Chapter 2, Section 4, first half of Sect. 5

2. **Homework 6:**

   Chapter 2, second half of 5; Section 6

Week 2a (continued): $A\vec{x} = \vec{0}$, homog.; Matrix Inverse.

The **Rank** of a matrix is defined to be
the number of non-zero rows in the RREF of
the matrix. We do not necessarily need the
RREF or even a REF to find the rank; for
example, a (square) upper-triagular matrix
that is $n \times n$ with non-zero diagonal entries
always has rank n. (why?)

We will discuss the results of the qualitative
theory more later on the text; but the first main
result is that when $A\vec{x} = \vec{b}$, with A an
$m \times n$ matrix, has rank(A) = rank$(A^{\#}) = n$,
the system has a unique solution.

(We're using $A^{\#} = (A|\vec{b})$ for the augmented matrix.)

Next, whenever rank(A) = rank$(A^{\#})$ we may
read-off a particular solution $\vec{x}_p$, so
that the system is consistent; while

rank$(A) \neq$ rank$(A^{\#})$ occurs only
when rank$(A^{\#})$ = rank(A) $+1$, in which
case the last equation reads $0 = 1$, which
is inconsistent, so the system has no solution.

Finally, if $r = \text{rank}(A) = \text{rank}(A^{\#}) < n$, there are infinitely many solutions; and with r leading 1's; so r bound variables, there are $d = n - r$ free varibles and d linearly independent solutions to the homog. equation.

We have one more case for matrix multiplication,

3. (a) $m \times n$ matrix A by $q \times p$-matrix B, only when $n = q$, is the $m \times p$ matrix with columns

$$AB = A\left(\vec{b}_1, \ldots, \vec{b}_p\right)$$
$$= \left(A\vec{b}_1, \ldots, A\vec{b}_p\right).$$

3. (b) An alternate descrition giving the i, jth entry of AB : ((ith row of A) $\cdot$ (jth column of B))

Linear Combination property of matrix mult: we have another version of the 2nd case, above.

If $\vec{c}$ is the column vector with entries $c_1, c_2, \ldots, c_n$, we can also write the matrix product using columns of A as $A\vec{c} = (\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)\,\vec{c}$
$= c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n$. (why?)

Any sum of scalar multiples $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$ is called a **linear combination** of the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$.

A square matrix is said to be a **diagonal** matrix if the only non-zero entries are on the main diagonal (top-left to lower-right).

The n-by-n **identity matrix** $I = I_n$

is the diagonal matrix with all diagonal entries 1's.

If A is any n-by-n matrix,

we have $A \cdot I = I \cdot A = A$, so I behaves like the number 1 under multiplication (that is, $1a = a1 = a$, for all real a.).

Finally, A is called **invertible** or **non-singular** if the matrix equation $AX = XA = I$, has an n-by-n solution X, in which case we write $X = A^{-1}$, and call X the **inverse** of A.

An essential property of the inverse is that when it exists, it is unique. If we translate the matrix equation $AX = I$ into n systems

of equations for the columns of $X = (\vec{x}_1 \ldots \vec{x}_n)$,

we get $AX = A(\vec{x}_1 \ldots \vec{x}_n) = (A\vec{x}_1 \ldots A\vec{x}_n) = (\vec{e}_1, \ldots \vec{e}_n)$,

$[(A\vec{x}_1 \ldots A\vec{x}_n) = (\vec{e}_1, \ldots \vec{e}_n),]$ where $\vec{e}_j$ is the jth column of the identity matrix, so $A\vec{x}_j = \vec{e}_j$.

Since the inverse is unique, the system for each column has a unique solution, so A has rank n and the RREF of A is I.

Applying row reduction to the "augmented" matrix $(A|I)$, we get $(I|X)$, with $X = A^{-1}$.

For an example of a 3-by-3 inverse, we compute the inverse

of the matrix in **Example** $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{pmatrix}$.

Solution:

$$(A|I) = \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 2 & -3 & 3 & | & 0 & 1 & 0 \\ 1 & -1 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & -1 & 0 & 1 \end{pmatrix} (r_2 \to r_2 - 2r_1, r_3 \to r_3 - r_1)$$

$$\to \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{pmatrix} (r_3 \to -r_3)$$

$$\to \begin{pmatrix} 1 & -1 & 0 & | & -1 & 0 & 2 \\ 0 & -1 & 0 & | & -1 & 1 & -1 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{pmatrix} (r_2 \to r_2 + r_3, r_1 \to r_1 - 2r_3)$$

$$\to \begin{pmatrix} 1 & -1 & 0 & | & -1 & 0 & 2 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{pmatrix} (r_2 \to -r_2)$$

$$\to \begin{pmatrix} 1 & 0 & 0 & | & 0 & -1 & 3 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{pmatrix} (r_1 \to r_1 + r_2), \text{ so } A^{-1} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$