

Week 1:

Intro, Slope Fields, verify solution

1.4 Separable DE, Cooling

1.5 population

1.6 Linear Equations

Next Week: 1.7 Mixing

Problem:

Inflamable substance, temp $T_0 = 50$ (F), placed in hot oven, temp $T_m = 450$ (F). After 20 min substance temp $T = 150$. Find temp at 40 min. If substance ignites at 350, find time of combustion.

Solution:

Newton's Law of cooling: $T = T(t)$ temp at time t ,

$$\frac{dT}{dt} = -k(T - T_m), \quad T_m = 450, \quad t \text{ in min.}$$

$$\frac{dT}{dt} = -k(T - T_m), \quad T_m = 450, \quad T(0) = 50, \quad T(20) = 150,$$

find $T(40)$ and t_c so $T(t_c) = 350$. **Method:** separation.

$$\frac{dT}{T - T_m} = -k dt, \quad (T \neq T_m).$$

integral: $\ln |T - T_m| = -k t + c, \quad T_m = 450.$

Initial Data: $\ln(450 - T) = -k t + c,$

$$T(0) = 50, \quad T(20) = 150.$$

When $t = 0$, $\ln 400 = c$, $e^c = 400$. So $450 - T = e^{-kt} e^c$
 $= 400e^{-kt}$, and $T = T(t) = 450 - 400(e^{-k})^t$.

Next, when $t = 20$, $150 = 450 - 400(e^{-k})^{20}$,

$$\text{so } -300 = -400(e^{-k})^{20}, \quad e^{-k} = \left(\frac{3}{4}\right)^{\frac{1}{20}},$$

$$\text{and } T(t) = 450 - 400\left(\frac{3}{4}\right)^{\frac{t}{20}}.$$

Finally, $T(t) = 450 - 400\left(\frac{3}{4}\right)^{\frac{t}{20}}$, gives

$$T(40) = 225, \text{ and } T(t_c) = 350 \text{ gives } t_c = 96.4 \text{ minutes.}$$

(why?)

For $t = 40$, $\frac{t}{20} = 2$, then $(\frac{3}{4})^2 = \frac{9}{16}$ and

$$400 * \frac{9}{16} = 25 \cdot 9 = 225, \text{ so } T(40) = 450 - 225 = 225.$$

Next, for $T := t \rightarrow 450 - 400 * ((3/4)^{(t/20})$); Maple's

`fsolve(T(t)=350,t);` gives $t_c = 96.3768$.

Problem: A 200L tank is half full of a solution containing 100g of a dissolved chemical. A solution containing 0.5 g/L of the same chemical is pumped into the tank at a rate of 6 L/min. The well-stirred mixture is pumped out at a rate of 4 L/min. Determine the concentration of the chemical in the tank just before overflow.

Solution:

$V = V(t)$ and $A = A(t)$ will be the volume of the solution in the tank (in L) and amount of chemical in the tank (in g), at time t (in min.). The “rate in” is, $r_1 = 6$ and “rate out”, $r_2 = 4$; and the “concentration in” is, $c_1 = 0.5$, while we solve for the “concentration out”, $c_2 = \frac{A}{V}$.

First, the initial volume $V_0 = 100$ (one half of the tank's volume), so $V(t) = 100 + (r_1 - r_2)t = 100 + 2t$ (“rate in” - “rate out”). Next, the tank overflows when V is 200, which occurs when $t = 50$, (why?) so we're looking for $c_2(50)$. Now our main DE says that the rate of change of the amount of the

chemical, $\frac{dA}{dt}$, is the difference $r_1 c_1 - r_2 c_2$

(“rate in” - “rate out”), which we re-write as

$$\frac{dA}{dt} + \left(\frac{4}{100 + 2t} \right) A = 3, \text{ (using } c_2 = \frac{A}{V} \text{)}.$$

Now $\frac{dA}{dt} + \left(\frac{4}{100 + 2t}\right) A = 3$ is a Linear DE,

with coefficient $P = \frac{4}{100 + 2t} = \frac{2}{50 + t}$.

So $\int P dt = 2 \ln(50 + t) = \ln(50 + t)^2$,

and the integral factor $I = e^{\int P dt} = e^{\ln(50+t)^2}$
 $= (50 + t)^2$. Multiplying both sides by I and using
the Main Property gives

$$\frac{d}{dt} ((50 + t)^2 A) = 3(50 + t)^2,$$

then integrating gives $(50 + t)^2 A = (50 + t)^3 + c$, so

$$A = 50 + t + \frac{c}{(50 + t)^2}.$$

One last bit of data, the initial amount of the chemical
was 100g, which gives the initial condition $A(0) = 100$,

so $100 = 50 + \frac{c}{50^2}$, and $c = 50^3$,

$$A(t) = 50 + t + \frac{50^3}{(50 + t)^2}.$$

So from $A(t) = 50 + t + \frac{50^3}{(50 + t)^2}$, we have

$$c_2(50) = \frac{A(50)}{V(50)} = \frac{100 + \frac{125000}{100^2}}{200}$$

$$= \frac{100 + \frac{125}{10}}{200} = \frac{100 + \frac{25}{2}}{200}$$

$$= \frac{\frac{225}{2}}{200} = \frac{225}{400} = \frac{9}{16} \text{ g/L for the}$$

concentration at the time of overflow.

Population [off homework!]

Logistic model: $P = P(t)$ pop. at time t ,

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P \quad \text{where } C \text{ is the carrying capacity}$$

(asymptotic limit (max) population), and r is the birth rate per individual; t in years.

Key Step in method: After $C dP = r(C - P)P dt$,

separation gives $\frac{C dP}{(C - P)P} = r dt$,

where we use Partial Fractions to get

$$\frac{C}{(C - P)P} = \frac{A}{P} + \frac{B}{C - P}$$

with $A = B = 1$, so $\int \left(\frac{1}{P} + \frac{1}{C - P} \right) dP = rt + c_1$,

where the integral is $\ln P - \ln(C - P) = \ln \frac{P}{C - P}$.

Not-so Key Steps: Raising to e gives $\frac{P}{C - P} = e^{rt} e^{c_1}$,

where we replace e^{c_1} by c_2 , and solve for P , then

use the initial condition to replace c_2 by $\frac{P_0}{C - P_0}$

and clear fractions to get the solution in a

standard form.

Problem Setup: In practice, we don't

use formulas, and simply observe that

if the maximum population, the carrying capacity,

is everyone, say $C = 1500$, and the DE is

$\frac{dP}{dt} = kP(1500 - P)$, we solve using

Partial Fractions, without needing any other formulas.