Week 9:

1. 4.3 Undetermined Coef
2. 4.5 Springs/Circuits

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Problem 3.

Solve $y'' - 4y' + 3y = 6e^{2x}$.

Solution:

(a) Find the general solution y_H of the homogeneous equation $y'' - 4y' + 3y = 0$.

(a) $y_H = c_1 e^x + c_2 e^{3x}$.

(b) Determine the value of the constant A_0 so $y_p = A_0 e^{2x}$ is a particular solution.

$y_p = A_0 e^{2x}, y_p' = A_0(2e^{2x}), y_p'' = 4A_0 e^{2x}$, so

$$6e^{2x} = 4A_0 e^{2x} - 4(2A_0 e^{2x}) + 3(A_0 e^{2x})$$
$$= -A_0 e^{2x}, \text{ so } A_0 = -6, y_p = -6e^{2x}.$$

(c) Determine the general solution.

$y = y_H + y_p = c_1 e^x + c_2 e^{3x} - 6e^{2x}$.

Compare 3d: If $F = 3x^2$, find constants a_0, a_1 and a_2

so $y_p = a_0 + a_1 x + a_2 x^2$.

$y_p' = a_1 + 2xa_2$, $y_p'' = 2a_2$ gives polynomial equation

$3x^2 = 2a_2 - 4(a_1 + 2xa_2) + 3(a_0 + a_1 x + a_2 x^2)$ all x.

x^2**-coef:** $3 = 3a_2$, so $a_2 = 1$.

x**-coef:** $0 = -4(2a_2) + 3a_1$ with $a_2 = 1$, so $a_1 = \frac{8}{3}$.

constant term: $0 = 2a_2 - 4(a_1) + 3(a_0)$, so $a_0 = \frac{26}{9}$.

So $y_p = \frac{26}{9} + \frac{8}{3}x + x^2$.

We have one more **linearity property:**

If $y'' + a_1 y' + a_2 y = F_1$ and $y'' + a_1 y' + a_2 y = F_2$

have particular solutions y_{p1} and y_{p2}, then

$y'' + a_1 y' + a_2 y = F$, for $F = F_1 + F_2$, has

particular solution $y_p = y_{p1} + y_{p2}$.

Example: $y'' - 4y' + 3y = 6e^{2x} + 3x^2$ has

solution $y_p = -6e^{2x} + \frac{26}{9} + \frac{8}{3}x + x^2$.

Note: There is a problem on the HW that

is easier when done using this method. (Which?)

Problem 4:

Solve $y'' + 4y' + 5y = 24 \sin x$.

Solution:

The usual trial function for $F(x) = 24 \sin x$ is $y_p = A_0 \cos x + B_0 \sin x$. Before attempting to compute y_p' and y_p'' to find A_0 and B_0 (the "undetermined coefficients"), we check y_H for repetition. The homogeneous equation $y'' + 4y' + 5y = 0$ has characteristic polynomial $P(r) = r^2 + 4r + 5$. The quadratic formula gives roots $-2 \pm i$, say $r_1 = -2 + i$ and $r_2 = -2 - i$ (why?). But the "driving force" $F(x) = 24 \sin x$ occurs as a solution of the DE with roots $\pm i$, that is, $a \pm bi$ with $a = 0, b = 1$, which do not coincide with r_1, r_2 : the **usual** $y_p = A_0 \cos x + B_0 \sin x$ is the correct choice.

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Now with $y_p = A_0 \cos x + B_0 \sin x$, we have

$y_p' = -A_0 \sin x + B_0 \cos x$, and

$y_p'' = -A_0 \cos x - B_0 \sin x$. Substitution in the DE gives $24 \sin x = y_p'' + 4y_p' + 5y_p$

$$= (-A_0 \cos x - B_0 \sin x) + 4(-A_0 \sin x + B_0 \cos x) + 5(A_0 \cos x + B_0 \sin x).$$

Equating like terms

coef of $\cos x : -A_0 + 4B_0 + 5A_0 = 0$, so $B_0 = -A_0$.

coef of $\sin x : -B_0 + 4(-A_0) + 5B_0 = 24$,

so $-4A_0 + 4(B_0) = -8A_0 = 24$, gives $A_0 = -3, B_0 = 3$, then $y_p = -3 \cos x + 3 \sin x$, and

$y = y_H + y_p = e^{-2x}(c_1 \cos x + c_2 \sin x) - 3 \cos x + 3 \sin x.$

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Preliminaries on applications:

The above solution $y = y_H + y_p =$

$$e^{-2t}(c_1 \cos t + c_2 \sin t) - 3\cos t + 3\sin t$$

has a typical property found in our applications to springs and circuits. For physical reasons, y_H usually has an exponential decay as $t \to \infty$, so y_H is referred to as the **transient solution** and we get $y = y_H + y_p \to y_p$ as $t \to \infty$. We call y_p the **steady-state solution** because of this property.

Question: How does the behavior of y_p differ in the usual trial solution, as compared with the modified solution? (resonance)

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Problem 4b:

Solve $y'' + 16y = 24\cos(4x)$.

Solution:

There are several problems here. Taking $y_p = A\cos(4x)$ may not be enough, as $y_p' = -4A\sin(4x)$. So we might try $y_p = A\cos(4x) + B\sin(4x)$. While that would be the correct trial function for a problem like Solve $y'' + 2y' + 16y = 24\cos(4x)$, it doesn't work here. The reason: the above equation has general homog solution $y_H = c_1 \cos 4x + c_2 \sin 4x$, so for any A and B we'd have $y_H + y_p = y_H$, with no new independent solutions.

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Let's try again. **Problem 4b:**

Solve $y'' + 16y = 24\cos(4x)$.

Solution:

Take $y_p = x(A\cos 4x + B\sin 4x)$.

So $y_p' = (A\cos 4x + B\sin 4x) + x(-4A\sin 4x + 4B\cos 4x)$,

and $y_p'' =$

$= 2(-4A\sin 4x + 4B\cos 4x) + x(-16A\cos 4x - 16B\sin 4x)$,

So "equating like terms" in

$24\cos 4x = y_p'' + 16y_p$

$= [2(-4A\sin 4x + 4B\cos 4x)+$

$x(-16A\cos 4x - 16B\sin 4x)] + 16(x(A\cos 4x + B\sin 4x)$

gives $24 = 8B, 0 = -8A$. So the above $y_p = 3x\sin(4x)$.

Recall **General Solution:** (2) to solve a 2nd order non-homogeneous 2nd linear DE $y'' + a_1y' + a_2y = F$,

find a particular solution y_p, then the general solution is $y = y_H + y_p$, where y_H is the general solution of the homogeneous equation (same coef, $F = 0$).

Also recall that we have a collection of functions that occur as solutions of (second order?) homogeneous DE:

(1) e^{ax}; (2) xe^{ax}; (3) x^2 (for $a = 0, y''' = 0$);

or, (4) $e^{ax}\cos(bx)$ or $e^{ax}\sin(bx)$.

In the **method of undetermined coefficients,**

we have developed "trial functions" for y_p when

F is obtained from the functions (1) - (4), and in each case differentiate, twice, plug the result into the non-homog DE, and solve (**"determine!"**) the coefficents.

The general homogeneous mass-spring system is

$$m\frac{d^2y}{dt^2} + f\frac{dy}{dt} + ky = 0,$$

where m is the mass, k is the spring constant and f is the friction constant. In the case $f = 0$ the system is said to be **undamped.**

When $f \neq 0$ the system is **damped,** and the motion depends primarily on the type of damping, determined by the roots.

Problem 5 Solve $y'' + 2y' + 5y = 0$, with IV $y(0) = 1, y'(0) = 3$, and determine whether the system is **overdamped, critically damped** or **damped and oscillating,** describe the sketch. (Terms from 4.5.15.)

(soln.) char. polyn. $r^2 + 2r + 5 = 0$, roots are $-1 \pm 2i$, complex, so the system is underdamped. With $y = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$ we get $1 = y(0) = c_1$,

$y = e^{-t}(\cos 2t + c_2 \sin 2t)$, so

$y' = -e^{-t}(\cos 2t + c_2 \sin 2t) + e^{-t}(-2\sin 2t + 2c_2 \cos 2t)),$ (continued ...)

$3 = y'(0) = -(1 + 0) + (0 + 2c_2)$, so $c_2 = 2$,

$y = e^{-t}(\cos 2t + 2\sin 2t)$.

(graphing observations: see 3 early max/mins,

2 equilibrium crosses, exponential decay.)

Problem 6:

Solve $y'' + 3y' + 2y = 3\cos t, y(0) = 2, y'(0) = -1$.

Solution:

$y_p = A_0 \cos t + B_0 \sin t$, since y_H is given by

roots $r = -1, -2$. The motion in the homog. system

is overdamped. We have $y_p' = -A_0 \sin t + B_0 \cos t$;

$y_p'' = -A_0 \cos - B_0 \sin t$,

So $3\cos t = (-A_0 \cos - B_0 \sin t) + 3(-A_0 \sin t + B_0 \cos t)$

$+2(A_0 \cos x + B_0 \sin x)$. Then $0 = -B_0 - 3A_0 + 2B_0$

gives $B_0 = 3A_0$; $3 = -A_0 + 3B_0 + 2A_0 = A_0 + 3(3A_0)$,

so $A_0 = \frac{3}{10}$, $B_0 = \frac{9}{10}$,

$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{3}{10}\cos t + \frac{9}{10}\sin t$.

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3 types, many cases:

(a) homogeneous, no damping or un-damped:

Amplitude, frequency ...

(b) homogeneous, with damping:

underdamping (complex roots, oscillating), critical damping

(repeated real), overdamping (distinct real)

(c) non-homogeneous: $y = y_H + y_p$ where (with damping)

$y_H \to 0$ as $t \to \infty$ is the **transient solution,**

and $y = y_H + y_p \to y_p$ as $t \to \infty$, so y_p is the

steady-state solution.

Finally, a case with no damping:

Problem 7 Solve $y'' + 4y = 0, y(0) = 2, y'(0) = 4$.

(Soln.) char. polynomial $r^2 + 4$, roots $\pm 2i$, general soln

$y = c_1 \cos 2x + c_2 \sin 2x$; so $2 = y(0) = c_1$,

$y = 2\cos 2x + c_2 \sin 2x$. Then

$y' = -4\sin 2x + 2c_2 \cos 2x$, so $4 = y'(0) = 2c_2$, gives

$c_2 = 2$, and IVP soln $y = 2\cos 2x + 2\sin 2x$.