

Week 4:

2.1 \mathbf{R}^n and Vector Spaces

2.2 Subspaces/Spanning

2.3 Independence/Bases

2.4 Nullspaces

1. Vector addition; scalar multiplication

in \mathbf{R}^2 . Vectors are $\vec{x} = (x, y)$, with x, y real numbers.

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

parallelogram law:

$$(2, 1) + (1, 4) = (3, 5) \text{ (picture!)}$$

$k(x, y) = (kx, ky)$, scaling factor

$$3(2, 1) = (6, 3); \quad -\frac{1}{2}(2, 1) = (-1, -\frac{1}{2}). \text{ (sketch)}$$

zero vector: $\vec{0} = (0, 0)$. additive inverse: $-\vec{x} = -(x, y) = (-x, -y)$.

distributive rules (2)

standard unit vectors: $\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$.

Linear combination property: $\vec{x} = (x, y) = (x, 0) + (0, y) = x\vec{i} + y\vec{j}$.

\mathbf{R}^3 : $\vec{x} + \vec{y}$, $k \cdot \vec{x}$,

$$\vec{0} = (0, 0, 0); \quad -\vec{x} = -(x, y, z) = (-x, -y, -z).$$

standard unit vectors $\vec{i}, \vec{j}, \vec{k}$.

$$\vec{x} = (x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z) = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$\begin{aligned}\mathbf{R}^n : \vec{x} + \vec{y} &= (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) \\ &= (x_1 + y_1, \dots, x_n + y_n).\end{aligned}$$

$$k \cdot \vec{x} = k(x_1, x_2, \dots, x_n) = (kx_1, \dots, kx_n)$$

$$\begin{aligned}\vec{0} &= (0, 0, \dots, 0), \quad -\vec{x} = -(x_1, x_2, \dots, x_n) \\ &= (-x_1, -x_2, \dots, -x_n).\end{aligned}$$

V any collection, elements $\vec{v} \in V$
called **vectors**.

Formula for vector plus and scalar mult.

Usually scalars are real $k \in \mathbf{R}$;

may also have $k \in \mathbf{C}$, complex.

$(V, +, \cdot)$ is a **Vector Space** if the

vector $+$ and scalar mult satisfy 10 rules:

2 closure rules

4 rules for $+$ (including $\vec{0}$; and $-\vec{x}$)

2 rules for \cdot (including $1 \cdot \vec{v} = \vec{v}$)

2 distributive laws. (2+4+2+2 = 10 or 2+8.)

Examples: \mathbf{R}^n ; $V = M_{2 \times 2}(\mathbf{R})$;

$F(a, b) =$ functions: $f+g$, $k \cdot f$, 0 , $-f$.

$P_k =$ polynomials with real coef. degree $< k$.

Problem

Is $P =$ polynomials of degree exactly 2
a vector space?

Solution. (closure?), zero vector?

(try $p_1(x) + p_2(x) = (x^2 + 3x) + (-x^2 - 2)$.)

Problem Express the solutions of

$A\vec{x} = \vec{0}$ as a subset of \mathbf{R}^4

$$\text{for } A = \begin{pmatrix} 1 & 3 & -2 & 1 \\ 3 & 10 & -4 & 6 \\ 2 & 5 & -6 & -1 \end{pmatrix}$$

Solution:

$$A \rightarrow \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -8 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$= A_R$ in Reduced Row Echelon Form (RREF).

Last Steps: The bound variables are x_1 and x_2

(since there are leading 1's in col 1 and in col 2);

so $x_3 = s$ and $x_4 = t$ are the free variables.

Use the i th row of A_R to solve for the i th bound variable:

$$x_1 - 8x_3 - 8x_4 = 0, \text{ so } x_1 = 8s + 8t, \text{ and}$$

$$x_2 + 2x_3 + 3x_4 = 0, \text{ so } x_2 = -2s - 3t; \text{ so}$$

$$(x_1, x_2, x_3, x_4) = (8s + 8t, -2s - 3t, s, t), \text{ for } s, t \in \mathbf{R}.$$

Note 1: We reduce A rather than $A^\# = (A|\vec{0})$,

but the equations correspond to the augmented matrix $(A_R|\vec{0})$.

Note 2: (Pre-view of 2.4) Using vector addition and scalar mult

in \mathbf{R}^4 , we say that the solutions (“Nullspace”) are “spanned” by the

coefficient vectors $(8, -2, 1, 0)$ and $(8, -3, 0, 1)$

and that these two vectors are a “spanning set”

for the solution space. Properties of the free variables x_3 and x_4 will show that these two vectors are “independent”, so we say that these two vectors are a “basis” for the solutions.

So in the homogeneous system $A\vec{x} = \vec{0}$, with $(x_1, x_2, x_3, x_4) = (8s + 8t, -2s - 3t, s, t)$, for $s, t \in \mathbf{R}$,

we find the coefficient vector of each parameter

$$\begin{aligned}\vec{x} &= (8s, -2s, s, 0) + (8t, -3t, 0, t) \\ &= s(8, -2, 1, 0) + t(8, -3, 0, 1), \text{ which}\end{aligned}$$

expresses each solution as a linear

combination of the vectors $\vec{u}_1 = (8, -2, 1, 0)$

and $\vec{u}_2 = (8, -3, 0, 1)$, so we say that \vec{u}_1 and \vec{u}_2 **span**

the vector space of solutions; or that $\{\vec{u}_1, \vec{u}_2\}$ is a **spanning set** for the space.

2. Subspaces and Spanning Sets

Recall that a Vector Space V may be any collection with formulas for vector plus and scalar mult, $V = (V, +, \cdot)$ with 10 rules.

In practice, in Math 205, Vector Spaces will usually be subsets of one of four standard

Examples: \mathbf{R}^n ; $V = M_{m \times n}(\mathbf{R})$;

$F(a, b) =$ functions: $f+g$, $k \cdot f$, 0 , $-f$.

$P_n =$ polynomials with real coef. degree $< n$.

We usually assume the 10 rules for these four vector spaces (the rules are easier to verify than to remember). If S is a subset of one of these standard spaces V , and we want S itself to be a vector space using the same formulas for $+$ and scalar mult as in V ,

eight of the rules in S follow from the corresponding rules in V . For example, if every vector in S is in V , and

$\vec{u}, \vec{v} \in S$, then

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$ since $\vec{u}, \vec{v} \in V$ and this

is one of the 10 rules we're assuming for our known, standard example V .

So to establish that our collection S is a vector space

we only need to check the last two rules, in

which case we say that S is a

Subspace of V . The

two subspace conditions are the

closure rules (1) for every

$\vec{u}, \vec{v} \in S$, check $\vec{u} + \vec{v} \in S$ (closure under vector $+$),

and (2) for every $\vec{u} \in S$, and every scalar

$c \in \mathbf{R}$, check $c\vec{u} \in S$ (closure under scalar mult).

Checking that S is NOT a subspace is

easy - for example, if S is all $(x, y) \in \mathbf{R}^2$ so that $x + y = 1$,

we have $y = 1 - x$, for any real x and

so $(1, 0), (0, 1) \in S$ (taking $x = 1, x = 0$).

But $(1, 0) + (0, 1) = (1, 1)$ is not in S (why? check sketch of S).

So this subset is not a subspace of \mathbf{R}^2 .

We don't need any other reason, but can

also check closure under scalar mult.

Since $c\vec{u}$ must be in S for every

scalar, we must have, for $c = 0$, $0 \cdot \vec{u} = \vec{0}$, with $\vec{0} = (0, 0)$ in S .

But $(0, 0)$ is not in S (sketch!).

By contrast $T = \text{all } (x, y) \in \mathbf{R}^2$ so

that $x + y = 0$, is a subspace (check).

3. Monday's notes.

For the formal definition of **span** and **spanning set** we take

vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ in a vector space V . For the most part, we think of V as being (1) \mathbf{R}^n ; (2) $F(a, b)$; or, (3) a subspace of either (a) Euclidean n -space or (b) of k -differentiable functions.

A vector $\vec{v} \in V$ is called a

linear combination of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ if there are

scalars c_1, c_2, \dots, c_k so that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$.

The **span** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ is the collection of all $\vec{v} \in V$ so that \vec{v} is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$.

The Span is a subspace of V . If $\text{Span} = V$,

we say that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ is a **spanning set** of V .

Examples: (1) \vec{i}, \vec{j} span R^2 ;

(2) $\{\vec{i}, \vec{j}, \vec{k}\}$ is a spanning set for R^3 ;

(3) the coef vectors obtained from the RREF of A span the solutions of $A\vec{x} = \vec{0}$.

Problem

Does the vector $\vec{v} = (3, 3, 4)$ belong to the subspace $\text{Span}(\vec{v}_1, \vec{v}_2)$ of \mathbf{R}^3 , for $\vec{v}_1 = (1, -1, 2), \vec{v}_2 = (2, 1, 3)$?

Solution

$$\begin{aligned} \text{We have } \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

Observe that this is the linear system

$$\text{with augmented matrix } (\vec{v}_1 \vec{v}_2 | \vec{v}) = \left(\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 1 & 3 \\ 2 & 3 & 4 \end{array} \right).$$

$$\text{Row reduction gives } \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{array} \right).$$

We see that this is a consistent system

($2 = r = r^\# = 2$), so YES, \vec{v} is in the span.

We are often asked to continue by finding an explicit linear combination, solving for c_1, c_2 .

For this, we continue to the RREF. We find

$$c_2 = 2, c_1 = -1, \text{ and } \vec{v} = -\vec{v}_1 + 2\vec{v}_2.$$