**Math 205, Fall 2009**

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**Week 2:** $A\vec{x} = \vec{0}$, homog.; Matrix Inverse.

The **Rank** of a matrix is defined to be
the number of non-zero rows in the RREF of the matrix. We do not necessarily need the RREF or even a REF to find the rank; for example, a (square) upper-triagular matrix that is $n \times n$ with non-zero diagonal entries always has rank $n$. (why?)

We will discuss the results of the qualitative theory more later on the text; but the first main result is that when $A\vec{x} = \vec{b}$, with $A$ an $m \times n$ matrix, has $\text{rank}(A) = \text{rank}(A^{\#}) = n$, the system has a unique solution.

(We're using $A^{\#} = (A|\vec{b})$ for the augmented matrix.)

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Next, whenever $\text{rank}(A) = \text{rank}(A^{\#})$ we may read-off a particular solution $\vec{x}_p$, so that the system is consistent; while

$\text{rank}(A) \neq \text{rank}(A^{\#})$ occurs only when $\text{rank}(A^{\#}) = \text{rank}(A) + 1$, in which case the last equation reads $0 = 1$, which is inconsistent, so the system has no solution.

Finally, if $r = \text{rank}(A) = \text{rank}(A^{\#}) < n$, there are infinitely many solutions; and with $r$ leading 1's; so $r$ bound variables, there are $d = n - r$ free varibles and $d$ linearly independent solutions to the homog. equation.

We have one more case for matrix multiplication,

3. (a) $m \times n$ matrix $A$ by $q \times p$-matrix $B$, only when $n = q$, is the $m \times p$ matrix with columns

$$AB = A\left(\vec{b}_1, \ldots, \vec{b}_p\right)$$
$$= \left(A\vec{b}_1, \ldots, A\vec{b}_p\right).$$

3. (b) An alternate descrition giving the $i, j$th entry of $AB$ : (($i$th row of $A$) $\cdot$ ($j$th column of $B$))

**Linear Combination property** of matrix mult: we have another version of the 2nd case, above.

If $\vec{c}$ is the column vector with entries $c_1, c_2, \ldots, c_n$, we can also write the matrix product using columns of $A$ as $A\vec{c} = (\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)\,\vec{c}$
$= c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n$. (why?)

Any sum of scalar multiples $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$ is called a **linear combination** of the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$.

A square matrix is said to be a **diagonal** matrix if the only non-zero entries are on the main diagonal (top-left to lower-right).

The $n$-by-$n$ **identity matrix** $I = I_n$ is the diagonal matrix with all diagonal entries 1's.

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If $A$ is any $n$-by-$n$ matrix, we have $A \cdot I = I \cdot A = A$, so $I$ behaves like the number 1 under multiplication (that is, $1a = a1 = a$, for all real $a$.).

Finally, $A$ is called **invertible** or **non-singular** if the matrix equation $AX = XA = I$, has an $n$-by-$n$ solution $X$, in which case we write $X = A^{-1}$, and call $X$ the **inverse** of $A$.

An essential property of the inverse is that when it exists, it is unique. If we translate the matrix equation $AX = I$ into $n$ systems of equations for the columns of $X = (\vec{x}_1 \ldots \vec{x}_n)$, we get $AX = A(\vec{x}_1 \ldots \vec{x}_n) = (A\vec{x}_1 \ldots A\vec{x}_n) = (\vec{e}_1, \ldots \vec{e}_n)$, [$(A\vec{x}_1 \ldots A\vec{x}_n) = (\vec{e}_1, \ldots \vec{e}_n)$,] where $\vec{e}_j$ is the $j$th column of the identity matrix, so $A\vec{x}_j = \vec{e}_j$.

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Since the inverse is unique, the system for each column has a unique solution, so $A$ has rank $n$ and the RREF of $A$ is $I$.

Applying row reduction to the "augmented" matrix $(A|I)$, we get $(I|X)$, with $X = A^{-1}$.

For an example of a 3-by-3 inverse, we compute the inverse

of the matrix in **Example** $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{pmatrix}$.

**Solution:**

$$(A|I) = \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 2 & -3 & 3 & | & 0 & 1 & 0 \\ 1 & -1 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & -1 & 0 & 1 \end{pmatrix} (r_2 \to r_2 - 2r_1, r_3 \to r_3 - r_1)$$

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$$\to \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{pmatrix} (r_3 \to -r_3)$$

$$\to \begin{pmatrix} 1 & -1 & 0 & | & -1 & 0 & 2 \\ 0 & -1 & 0 & | & -1 & 1 & -1 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{pmatrix} (r_2 \to r_2 + r_3, r_1 \to r_1 - 2r_3)$$

$$\to \begin{pmatrix} 1 & -1 & 0 & | & -1 & 0 & 2 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{pmatrix} (r_2 \to -r_2)$$

$$\to \begin{pmatrix} 1 & 0 & 0 & | & 0 & -1 & 3 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{pmatrix} (r_1 \to r_1 + r_2), \text{ so } A^{-1} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$