Week 6, 7:

2.5 Wronskian

3.1 Intro, Slope Fields, verify solution

3.2 Separable DE

3.4 Linear Equations

3.6 Cooling/Mixing

Problem 1:

Verify that the function $y = c_1\sqrt{x}$ is a solution of $y\prime = \dfrac{y}{2x}$

Solution:

Compute $y\prime$ and check.

$$y\prime = c_1(\tfrac{1}{2})x^{-\frac{1}{2}}.$$

$$\frac{y}{2x} = \frac{c_1\sqrt{x}}{2x}$$

$$= c_1(\tfrac{1}{2})\frac{\sqrt{x}}{(\sqrt{x})^2}$$

$$= c_1(\tfrac{1}{2})\frac{1}{\sqrt{x}}$$

$$= y\prime.$$

Problem 2:

Determine all values r so $y = e^{rx}$ is a solution to $y'' - 4y' + 3y = 0$.

Main Step: to solve linear DE

$$\frac{dy}{dx} + p(x)y = q(x)$$

multiply by the integral factor $f = e^{\int p\,dx}$

and use

$$\frac{d}{dx}(fy) = f\,(y' + py)$$

on the left to get $\dfrac{d}{dx}(fy) = fq$.

Problem:

Solve $\dfrac{dy}{dx} + \dfrac{2x}{(1-x^2)}y = 4x,\ -1 \le x \le 1.$

Solution:

Find integral factor, inside integral first:

$$\int \frac{2x}{(1-x^2)}\,dx$$

$$= -\ln(1-x^2) = \ln\left((1-x^2)^{-1}\right) \quad \text{(simplify!)}.$$

So $e^{\int \frac{2x}{(1-x^2)}\,dx} = e^{\ln\left((1-x^2)^{-1}\right)}$

$$= \left(1 - x^2\right)^{-1} = \frac{1}{1-x^2} = f.$$

Multiply by $f = \dfrac{1}{1-x^2}$ and use Main Property:

$$\frac{1}{1-x^2}\left(\frac{dy}{dx} + \frac{2x}{(1-x^2)}y\right) = \frac{4x}{1-x^2},$$

$$\frac{d}{dx}\left(\frac{y}{1-x^2}\right) = \frac{4x}{1-x^2}.$$

Now integration gives $\dfrac{y}{1-x^2} = (-2\ln(1-x^2)) + c,$

so $y = (1-x^2)\left(-\ln((1-x^2)^2) + c\right).$

Notice that we can check this instance of the

Main Property directly (using the product rule).

Problem:

Inflamable substance, temp $T_0 = 50$ (F), placed in hot oven, temp $T_m = 450$ (F). After 20 min substance temp $T = 150$. Find temp at 40 min. If substance ignites at 350, find time of combustion.

Solution:

Newton's Law of cooling: $T = T(t)$ temp at time t,

$$\frac{dT}{dt} = -k(T - T_m), \quad T_m = 450, \, t \text{ in min.}$$

$$\frac{dT}{dt} = -k(T - T_m),\ T_m = 450, T(0) = 50, T(20) = 150,$$

find $T(40)$ and t_c so $T(t_c) = 350$. **Method:** separation.

$$\frac{dT}{T - T_m} = -k\,dt, \quad (T \neq T_m).$$

integral: $\ln|T - T_m| = -k\,t + c,\ T_m = 450.$

Initial Data: $\ln(450 - T) = -k\,t + c,$

$T(0) = 50, T(20) = 150.$

When $t = 0$, $\ln 400 = c$, $e^c = 400$. So $450 - T = e^{-kt}e^c$

$= 400e^{-kt}$, and $T = T(t) = 450 - 400\,(e^{-k})^t$.

Next, when $t = 20$, $150 = 450 - 400\,(e^{-k})^{20}$,

so $-300 = -400\,(e^{-k})^{20}$, $e^{-k} = (\frac{3}{4})^{\frac{1}{20}}$,

and $T(t) = 450 - 400\,(\frac{3}{4})^{\frac{t}{20}}$.

Finally, $T(t) = 450 - 400\,(\frac{3}{4})^{\frac{t}{20}}$, gives

$T(40) = 225$, and $T(t_c) = 350$ gives $t_c = 96.4$ minutes.

(why?)

For $t = 40$, $\frac{t}{20} = 2$, then $(\frac{3}{4})^2 = \frac{9}{16}$ and

$400 * \frac{9}{16} = 25 \cdot 9 = 225$, so $T(40) = 450 - 225 = 225$.

Next, for $T := t \to 450 - 400 * ((3/4)^{(t/20)})$; Maple's

fsolve(T(t)=350,t); gives $t_c = 96.3768$.

Problem: A 200L tank is half full of a solution containing 100g of a desolved chemical. A solution containing 0.5 g/L of the same chemical is pumped into the tank at a rate of 6 L/min. The well-stirred mixture is pumped out at a rate of 4 L/min. Determine the concentration of the chemical in the tank just before overflow.

Solution:

$V = V(t)$ and $A = A(t)$ will be the volumn of the solution in the tank (in L) and amount of chemical in the tank (in g), at time t (in min.). The "rate in" is, $r_1 = 6$ and "rate out", $r_2 = 4$; and the "concentration in" is, $c_1 = 0.5$, while we solve for the "concentration out", $c_2 = \frac{A}{V}$.

First, the initial volumn $V_0 = 100$ (one half of the tank's volumn), so $V(t) = 100 + (r_1 - r_2)t = 100 + 2t$ ("rate in" - "rate out"). Next, the tank overflows when V is 200, which occurs when $t = 50$, (why?) so we're looking for $c_2(50)$. Now our main DE says that the rate of change of the amount of the chemical, $\frac{dA}{dt}$, is the difference $r_1c_1 - r_2c_2$ ("rate in" - "rate out"), which we re-write as

$$\frac{dA}{dt} + \left(\frac{4}{100 + 2t}\right) A = 3, \text{ (using } c_2 = \tfrac{A}{V}).$$

Now $\dfrac{dA}{dt} + \left(\dfrac{4}{100+2t}\right) A = 3$ is a Linear DE,

with coefficient $P = \dfrac{4}{100+2t} = \dfrac{2}{50+t}$.

So $\displaystyle\int P\,dt = 2\ln(50+t) = \ln(50+t)^2$,

and the integral factor $I = e^{\int P\,dt} = e^{\ln(50+t)^2}$

$= (50+t)^2$. Multiplying both sides by I and using the Main Property gives

$$\tfrac{d}{dt}\left((50+t)^2 A\right) = 3(50+t)^2,$$

then integrating gives $(50+t)^2 A = (50+t)^3 + c$, so

$$A = 50 + t + \frac{c}{(50+t)^2}.$$

One last bit of data, the initial amount of the chemical was 100g, which gives the initial condition $A(0) = 100$, so $100 = 50 + \dfrac{c}{50^2}$, and $c = 50^3$,

$$A(t) = 50 + t + \frac{50^3}{(50+t)^2}.$$

So from $A(t) = 50 + t + \frac{50^3}{(50+t)^2}$, we have

$$c_2(50) = \frac{A(50)}{V(50)} = \frac{100 + \frac{125000}{100^2}}{200}$$

$$= \frac{100 + \frac{125}{10}}{200} = \frac{100 + \frac{25}{2}}{200}$$

$$= \frac{\frac{225}{2}}{200} = \frac{225}{400} = \frac{9}{16}$$ g/L for the concentration at the time of overflow.

Logistic model: $P = P(t)$ pop. at time t,

$\frac{dP}{dt} = r(1 - \frac{P}{C})P$ where C is the carrying capacity (asymptotic limit (max) population), and r is the birth rate per individual; t in years.

Key Step in method: After $C\,dP = r(C-P)P\,dt$,

separation gives $\dfrac{C\,dP}{(C-P)P} = r\,dt$,

where we use Partial Fractions to get

$$\frac{C}{(C-P)P} = \frac{A}{P} + \frac{B}{C-P}$$

with $A = B = 1$, so $\displaystyle\int \left(\frac{1}{P} + \frac{1}{C-P}\right) dP = rt + c_1$,

where the integral is $\ln P - \ln(C-P) = \ln \dfrac{P}{C-P}$.

Not-so Key Steps: Raising to e gives $\dfrac{P}{C-P} = e^{rt}e^{c_1}$,

where we replace e^{c_1} by c_2, and solve for P, then

use the initial condition to replace c_2 by $\dfrac{P_0}{C-P_0}$

and clear fractions to get the solution in a

standard form.

Problem Setup: In practice, we don't

use formulas, and simply observe that

if the maximum population, the carrying capacity,

is everyone, say $C = 1500$, and the DE is

$\dfrac{dP}{dt} = kP(1500 - P)$, we solve using

Partial Fractions, without needing any other formulas.