Week 4: (continued)

2.2 Subspaces/Spanning (continued)

2.3 Independence/Bases

2.4 Nullspaces

Linear independence/Bases and Dimension

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in a vector space V are **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_k$ so $\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$, with at least one $c_j \neq 0$. Such a vector equation is said to be a **relation** of linear dependence among the $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. A collection of vectors for which the only solution of the vector equation is the **trivial solution,** $c_1 = 0, \ldots, c_k = 0$ is said to be **linearly independent.**

Examples: (1) $\vec{i}, \vec{j}$ in R^2; and (2) $\{\vec{i}, \vec{j}, \vec{k}\}$ in R^3; are linearly independent.

Verification: $c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (c_1, c_2, c_3)$

$= (0,0,0) = \vec{0}$, only when $c_1 = 0, c_2 = 0, c_3 = 0$.

Our main objective is the definition of basis. Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in a vector space V that are are both (1) a spanning set for V and (2) linearly independent are called a **basis** for V. So

(1) $\vec{i}, \vec{j}$ are a basis of R^2; and

(2) $\{\vec{i}, \vec{j}, \vec{k}\}$ is a basis of R^3.

A vector space has many bases, but the main fact is that if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is a basis of V and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_j$ is a second basis of V, then the number of vectors is the same: $k = j$. The number of vectors in a basis of V is called the **dimension** of V.

We start with verifications of linear independence in subspaces of $\mathbf{R}^n$

Problem

Determine whether the vectors $(1, 2, 3)$, $(1, -1, 2)$ and $(1, -4, 1)$ are LI or LD in $\mathbf{R}^3$.

If they are LD find a relation of linear dependence.

Solution [again; same method!]

Write the vector equation as an equation in column vectors, then

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -4 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} .$$

Observe that this is the homog linear system with coef matrix $A = (\vec{v}_1 \vec{v}_2 \vec{v}_3)$.

For n-by-n systems the system has a unique solution exactly when the determinant is non-zero.

We have $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -4 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -6 \\ 0 & -1 & -2 \end{vmatrix} = 0$

so the system has nontrivial solutions

and the vectors are LD. To find a

relation we continue the reduction

$$A \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

so a spanning vector is $(1, -2, 1)$,

and $\vec{0} = \vec{v}_1 - 2\vec{v}_2 + \vec{v}_3$

is a nontrivial relation of LD.

Example: Show that the vectors $\vec{v}_1 = (1, 1)$ $\vec{v}_2 = (1, -1)$ are a basis of $\mathbf{R}^2$.

Solution. We know that $\vec{i}, \vec{j}$ are a basis of $\mathbf{R}^2$; so that the dimension of $\mathbf{R}^2$ is $\ldots$.

We check that $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (1, -1)$ are linearly independent (LI) since $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ $= -2 \neq 0$. Checking that $\vec{v}_1$ and $\vec{v}_2$ span $\mathbf{R}^2$ by expressing each $\vec{v} = (x, y) \in \mathbf{R}^2$ as a linear combination (as we did for $\vec{i}, \vec{j}$)
takes a little bit of effort, and involves fractions.

Checking that each vector equation $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ has a
solution (without finding c_1, c_2) is easier. (why?)

But the main fact has a **practical version** that saves any computation, or even very much further thought. If $\vec{v}_1$ and $\vec{v}_2$ did not span $\mathbf{R}^2$ there would be a 3rd vector $\vec{v}_3$ not in the span of $\vec{v}_1, \vec{v}_2$.
But that would make $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent. We could consider adding further vectors, each new one giving a

larger collection of indep vectors; but we already have too many, as we've shown $\dim(\mathbf{R}^2) \geq 3$. So $\vec{v}_1, \vec{v}_2$ have to span (for free, given the main fact), so $\vec{v}_1, \vec{v}_2$ is a basis for $\mathbf{R}^2$.

This is the use of the text's

Theorem 2.12.(1) If the $\dim(V) = n$ for a vector space V, then any LI set of n vectors is a basis of V.