Math 205, Fall 2009

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Text - Peterson-Sochacki

Office hours: M,F: 12:20-1:20, W: 9:30 - 10:30

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1. Course Info

2. Homework 1:

1.1: 3, 4, 11, 16; 1.2: 20

Due Friday

An $m \times n$ **matrix** A is an array

with m horizontal rows; n vertical columns

i, jth entry $a_{i,j}$ in the ith row, and

jth column.

row vector or row n-vector, $\vec{a}$, is a $1 \times n$

matrix, just one row.

column vector or column n-vector, $\vec{b}$, is a

$n \times 1$ matrix, just one column.

Example 1. Give the rows and columns of

$$A = \begin{pmatrix} 2 & 10 & 6 \\ 5 & -1 & 3 \end{pmatrix}.$$

What are the entries $a_{1,2}, a_{2,1}, a_{3,1}, a_{1,3}$?

The matrix sum, $A + B$, is defined only when A and B have the same shape; and then the i, jth entry of $A + B$ is $a_{i,j} + b_{i,j}$, the sum of the i, jth entries of A and B.

The scalar multiple of the matrix A by the scalar (number!) c is the matrix with the same shape as A, but with i, jth entry of $ca_{i,j}$.

Matrix multiplication: 1. row n-vector by column m-vector, only when $n = m$, is the number

$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$

where the a_i are the entries of $\vec{a}$ and the b_i are the entries of $\vec{b}$.

2. $m \times n$ matrix A by p-column vector $\vec{b}$, only when $n = p$, is the m-column vector with ith entry (ith row of A) $\cdot \vec{b}$.

Problem 2. Multiply $A = \begin{pmatrix} -1 & 2 \\ 4 & 7 \\ 5 & -4 \end{pmatrix}$ by

$$c = \begin{pmatrix} 5 \\ -1 \end{pmatrix}.$$

Solution:

$$Ac = \begin{pmatrix} -1 & 2 \\ 4 & 7 \\ 5 & -4 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)(5) + 2(-1) \\ 4(5) + 7(-1) \\ 5(5) + (-4)(-1) \end{pmatrix} = \begin{pmatrix} -7 \\ 13 \\ 29 \end{pmatrix}.$$

Linear Systems

If A is an $m \times n$ matrix with entries $a_{i,j}$,
$\vec{x}$ is the n-column vector with entries $x_1, \ldots x_n$,
and $\vec{b}$ is the m-column vector with entries $b_1, \ldots b_m$,
the matrix equation $A\vec{x} = \vec{b}$ gives m equations,
each of the form (ith row of A) $\cdot \vec{x} = b_i$,
which is called a **linear system of m equations in n variables.**

In the equation $A\vec{x} = \vec{b}$, the matrix A is called the **coefficient matrix** of the system, and $\vec{x}$ and $\vec{b}$ are called the vector of unknowns and the right-hand side vector, respectively.

Our preliminary qualitative description of the solutions of these systems (in 2-space and 3-space) suggests three cases: (1) just one unique solution;

(2) no solutions; or (3) infinitely many solns.

We say that the system of equations is **consistent** if there is at least one solution; and **inconsistent** if there are no solutions.

Two systems with the same solutions are called **equivalent.**

Our objective is to replace a given system by the simplest possible equivalent system. For the calculations - we don't use the equations, but instead one more matrix $A^{\#} = (A|\vec{b})$, which is called the **augmented matrix** of the system.

Problem 3.

For the system

$$\begin{aligned} x + \ y + \ z - w &= 3, \\ 2x + 4y - 3z + 7w &= 2 \end{aligned}$$

determine the coef. matrix A, the right-hand side vector $\vec{b}$ and the augmented matrix $A^{\#}$.

Solve the system using elemenary row operations, and write the solution in vector form.

Solution:

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -3 & 7 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$A^{\#} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 2 & 4 & -3 & 7 & 2 \end{array}\right).$$

$$\left(A|\vec{b} \right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 2 & -5 & 9 & -4 \end{array}\right) \quad (r_2 \to r_2 - 2(r_1))$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 1 & -\frac{5}{2} & \frac{9}{2} & -2 \end{array}\right) \quad (r_2 \to \tfrac{1}{2} r_2)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & \frac{7}{2} & -\frac{11}{2} & 5 \\ 0 & 1 & -\frac{5}{2} & \frac{9}{2} & -2 \end{array}\right) \quad (r_1 \to r_1 - r_2)$$

The augmented matrix has been “reduced” to Reduced Row Echelon Form (RREF). For the last step, we notice “leading 1’s” in column 1 and 2, so we call the variables corresponding to those columns bound variables - x and y are bound. The other variables are free variables, and we use each row to solve for one bound variable.

$x + \frac{7}{2}z - \frac{11}{2}w = 5$, so

$x = 5 - \frac{7}{2}z + \frac{11}{2}w$ and

$y - \frac{5}{2}z + \frac{9}{2}w = -2$, so

$y = -2 + \frac{5}{2}z - \frac{9}{2}w$.

Finally, we replace the free variables by parameters, $z = r, w = s$, then $(x, y, z, w) =$

$(5 - \frac{7}{2}r + \frac{11}{2}s, -2 + \frac{5}{2}r - \frac{9}{2}s, r, s)$

$= (5, -2, 0, 0) + r(-\frac{7}{2}, \frac{5}{2}, 1, 0) + s(\frac{11}{2}, -\frac{9}{2}, 0, 1)$.

1.1 Gauss-Jordan Elimination

We simplify the information in a matrix by using Elementary Row Operations (ERO's). There are 3:

1. $E_I : r_i \leftrightarrow r_j$ means switch rows i and j.
2. $E_{II} : r_i \rightarrow kr_i$ means multiply row i by $k \neq 0$.
3. $E_{III} : r_i \rightarrow r_i, r_j \rightarrow (r_j + kr_i)$ means replace (row j) by the (linear) combination (row j) + k(row i), leaving row i unchanged.

We say that matrices A and B are **row equivalent** if there is a sequence of ERO's that starts with A and ends with B.

We say that a matrix is (row) reduced to simplest form if we have a matrix in reduced row echelon form (RREF) that is row equivalent. We may record the specific row operations, in the order used, that give the row reduction.

Problem 4. Use Gauss-Jordan Elimination to solve

$$\begin{aligned} 2x_1 - \ x_2 + 3x_3 - \ x_4 &= \ 3 \\ 3x_1 + 2x_2 + \ x_3 - 5x_4 &= -6 \\ x_1 - 2x_2 + 3x_3 + \ x_4 &= \ 6 \end{aligned}$$

Solution: We use the augmented matrix $A^{\#}$,

$$\text{with } A^{\#} = (A|\vec{b}) \rightarrow \begin{pmatrix} 1 & -2 & 3 & 1 & | & 6 \\ 3 & 2 & 1 & -5 & | & -6 \\ 2 & -1 & 3 & -1 & | & 3 \end{pmatrix},$$

$$\rightarrow \begin{pmatrix} 1 & -2 & 3 & 1 & | & 6 \\ 0 & 8 & -8 & -8 & | & -24 \\ 0 & 3 & -3 & -3 & | & -9 \end{pmatrix},$$

$$\rightarrow \begin{pmatrix} 1 & -2 & 3 & 1 & | & 6 \\ 0 & 1 & -1 & -1 & | & -3 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{(REF)},$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & -1 & | & -3 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{(RREF)}.$$

[to be continued ...]

Note 1: the matrix in the next-to-last step is in Row Echelon Form (REF), but not reduced. In Gaussian elimination we may stop at a REF and continue solving using back-substitution. For Gauss-Jordan elimination, the back-substitution steps are also done as row operations on the augmented matrix. Small systems with a unique solution may be done by Gaussian elimination without confusion, since the objective is to find numbers; Gauss-Jordan elimination will be the method-of-choice for finding generators when there are infinitely many solutions, and can always be used.

Last step in Gauss-Jordan: From the REF we identify **free variables** and **bound variables.**

Again x_1 and x_2, are bound, the other variables are free variables, and recalling

$$(A|\vec{b})_R = \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

the 1st row corresponds to the equation

$x_1 + x_3 - x_4 = 0$, which we solve for

$x_1 = -x_3 + x_4$ and the 2nd row gives

$x_2 - x_3 - x_4 = -3$, so $x_2 = -3 + x_3 + x_4$,

and $(x_1, x_2, x_3, x_4) = (-x_3 + x_4, -3 + x_3 + x_4, x_3, x_4)$.